#8

7/15/96

ACTS-96-046

Finite State Markov Models for Error Bursts on the ACTS Land Mobile Satellite Channel

Jeff Slack
Michael Rice
Department of Electrical & Computer Engineering
Brigham Young University
Provo, UT 84602

May 1996

Contents

List of Tables

List of Figures

Chapter 1

INTRODUCTION

Land mobile satellite channels are characterized by multipath fading and shadowing which generate error bursts in the demodulated data streams. The combination of interleaving and error control coding is used to combat the degradations in data quality caused by these error bursts. Proper design of interleavers and error control codes for such applications requires knowledge of the statistics which characterize these error bursts.

Markov chain models have been used extensively to model error bursts encountered on telephone lines [1]. However, limited work has been reported on the applicability of these models to other kinds of channels. In mobile wireless applications, the models have been used to characterize the received power levels of a particular channel [2, 3, 4]. However, there are no current applications using Markov chains to model the types of error bursts that occur over the land mobile satellite channel. This thesis will introduce the types of error sequences that occur on this type of channel and explore the applicability of different models to land mobile satellite wireless data.

1.1 Executive Summary

The error gap distribution and the block error statistics are used to characterize the error bursts of channels with memory. The purpose of this thesis is to examine the applicability of four well known finite state Markov chains to model synthetic error bursts produced from measured power data on the Land Mobile Satellite Channel. The selected Markov chains are generative [1] models which means:

- The transition probabilities produce a state sequence which maps to an error digit sequence or an error gap sequence in an attempt to simulate the transitions in real channel behavior from good to bad.

- The model can be used to derive relevant performance statistics after suitable parameterization.

The motivation for using generative models is that they represent real channels with relative simplicity. However, as the accuracy of representing the channel is required to increase, the models require more and more states, and therefore become computationally unwieldy and negate the purpose of using a model to represent the channel.

The first model presented is the Gilbert model [5] which can approximate either the error gap distribution for larger gaps or the block error probabilities. Neither curve is modeled simultaneously with the other.

The Elliott model [6] shows slight improvement over the Gilbert model in characterizing the block error probabilities given the error gap distribution.

The McCullough model [7] differentiates between random and bursty errors more than other models. It can model either block error probabilities or the error gap distribution for larger gaps. Neither curve is modeled simultaneously with the other.

With the Fritchman model [8], the error gap distribution is modeled more accurately than all previous models, however, the block error probabilities cannot be characterized for the simplified model. The fully connected model can characterize the block error probabilities. Neither curve is modeled simultaneously with the other.

All of the models show that the characteristics of the error bursts on the channel, the error gap distribution and the block error probabilities, can be represented with simple Markov models. However both characteristics cannot be modeled simultaneously with these models, suggesting more complex models may be needed for this purpose. The model that provides the best compromise in modeling both characteristics is the Elliott model.

1.2 Summary of Contributions

- The application of finite state Markov models used to model error bursts on phone lines are applied to the types of error bursts associated with the Land Mobile Satellite Channel and assessed.

- The error gap distribution and the block error probability curves are modeled well individually, but both are not modeled simultaneously.
- The Elliott model is determined to be a reasonable compromise in representing either the error gap distribution or the block error probabilities.

Chapter 2

DATA ACQUISITION

2.1 Overview

NASA's Advanced Communications Technology Satellite (ACTS) provides a stationary platform ideally suited to the measurement of mobile propagation effects at K-(20 GHz) and Ka-(30 GHz) bands. JPL[1] has developed a proof-of-concept breadboard mobile terminal system to operate in conjunction with ACTS at K/Ka-band called the ACTS Mobile Terminal (AMT) [9]. Field tests conducted during the first 7 months of 1994 using the AMT provide channel characterization data for the K-band land-mobile satellite channel.

2.2 Experiment Description

As depicted in Figure 2.1, the system is comprised of a bent pipe propagation link connecting terminals at fixed and mobile sites. The forward channel originated at the fixed station with a 29.634 GHz pilot tone. This pilot tone was received by ACTS, mixed to the down-link frequency of 19.914 GHz, and transmitted on the Southern California spot beam. The forward channel offered a composite carrier-to-noise ration C/N_0 of 55.63 dB-Hz [10] and was the basis for the K-band results.

The AMT is equipped with a small ($8'' \times 3''$) high-gain reflector antenna [11] which tracks the satellite in azimuth for a fixed elevation angle (46° for these experiments). The antenna is mechanically steered and acquires and tracks the satellite over the entire 360° of azimuth with a pointing error of less than 2°. Vehicle turn rates of up to 44° per second can be accommodated. The antenna has a receiver gain to system temperature ratio G/T of -6 dB/K over a bandwidth of 300 MHz. The 3 dB beam width is ±9° in elevation and ±6° in azimuth. The

[1] Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109

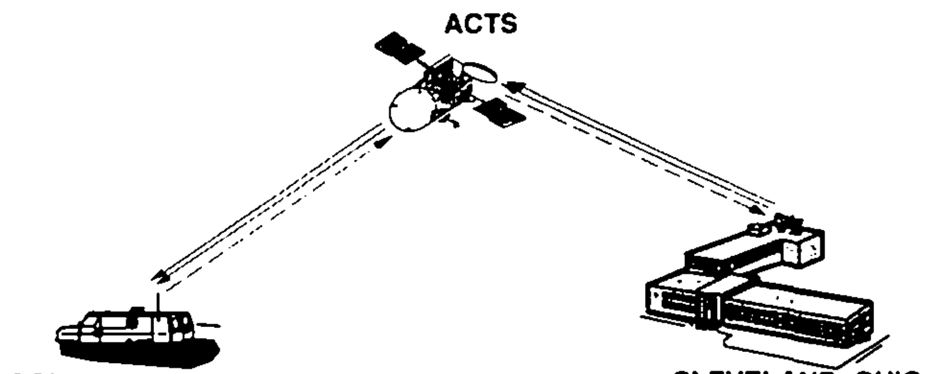

Figure 2.1: ACTS Mobile Terminal Communication Link

antenna pointing system enables the antenna to track the satellite for all practical vehicle maneuvers.

The Data Acquition System (DAS) illustrated in Figure 2.2 measures in-phase pilot voltage level and the non-coherent pilot power level. The in-phase pilot voltage level was sampled at 4000 samples/second in a bandwidth of 1.5 kHz and was used to analyze the channel characteristics presented in this thesis. The data were stored on 5 Gbyte Exabyte tapes for off-line evaluation. The vehicle position, vehicle velocity, and time stamp were derived from an on-board Global Positioning System (GPS) and updated once each second.

2.3 Data Classification

Data were collected in a variety of locations in order to characterize environments typical of mobile satellite applications. In the absence of any standard definitions for the various environmental conditions typical of land mobile satellite channels, a set of general classifications specific to Southern California was adopted. All runs in this measurement campaign were conducted in Pasadena, California which presents a seasonally invariant suburban environment. The environments are divided into three broad categories based on the type of roadway:

- **Category I:** a limited access multi-lane freeway

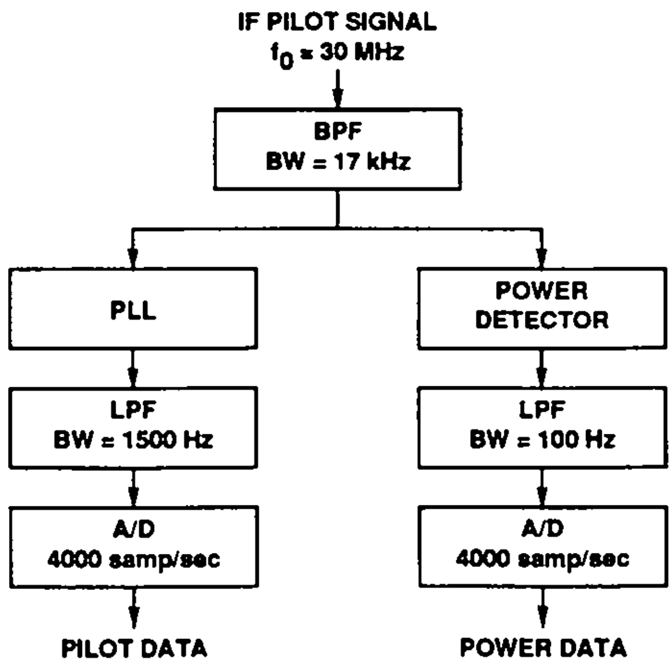

Figure 2.2: AMT data acquisition system block diagram.

- **Category II:** a broad suburban thoroughfare lined with trees and buildings. The tree canopies cause intermittent blockage and the buildings are either too far removed from the road side or not tall enough to cause significant blockage.

- **Category III:** a small, two-lane roadway lined with trees and buildings. The tree canopies often cover the entire road way and buildings are close enough to contribute to the fading process.

This description is most appropriate in this case since the type and kind of obstructions are strongly dependent on the nature of the road. Table 2.1 shows a summary of the environmental features of the AMT runs.

Table 2.1: Environmental Characteristics of AMT Propagation Runs

RUN	CAT.	DIRECTION		TERRAIN	OBSTRUCTIONS
020201	I	west	straight	hilly	none
070901	II	south, right lane	straight	flat	trees[2]
070903	III	north	curved	flat	trees[3]: canopies cover road
070905	II	north, right lane	straight	flat	trees[2]
070906	III	south	curved	flat	trees[3]: canopies cover road
070907	II	north, left lane	straight	flat	trees[2]
070912	III	south	curved	hilly	trees[4]: canopies cover road
070914	III	north	curved	hilly	trees[4]: canopies cover road
071016	II	north/south	curved	flat	trees[5]
071017	II	north/south	curved	flat	trees[5]
072405	II	east, left lane	straight	hilly	trees[6], utility poles
072406	II	west, left lane	straight	hilly	trees[6], utility poles
072407	II	east, right lane	straight	flat	trees[7], utility poles, bldgs
072408	II	west, right lane	straight	flat	trees[8], utility poles
072409	II	south, right lane	straight	flat	trees[2]
072410	II	north, right lane	straight	flat	trees[2]
072411	II	south, left lane	straight	flat	trees[2]
072412	II	north, left lane	straight	flat	trees[2]

2.4 Fade and Non-fade Events

Figure 2.3 shows the results for a typical run, in this case run 070906. In this figure the received voltage level is plotted as a function of the sample number. From the plot in Figure 2.3 it can be seen that the received voltage levels are not constant, but fluctuate rapidly about a certain voltage level λ, referred to as the mean received voltage level. The mean received voltage level is the voltage level that the receiver typically receives for a particular run when the path between

[2] In order of concentration: Southern Magnolia, Fan & Date Palm, Coastal Live Oak, California Pepper.

[3] In order of concentration: Coastal Live Oak, Southern Magnolias, Holly Oak.

[4] In order of concentration: Coastal Live Oak, Holly Oak, California Sycamore Deadora Cedar, California Pepper.

[5] In order of concentration: Oak, Pine, Sycamore, Magnolia, Cedar, Eucalyptus, Palm, California Pepper, Italian Cyprus.

[6] In order of concentration: Italian Cyprus, Palm, California Sycamore, Deadora Cedar.

[7] In order of concentration: Ficus (aka Indian Laurel Fig), Date Palm.

[8] In order of concentration: Eucalyptus, Fan and Date Palm.

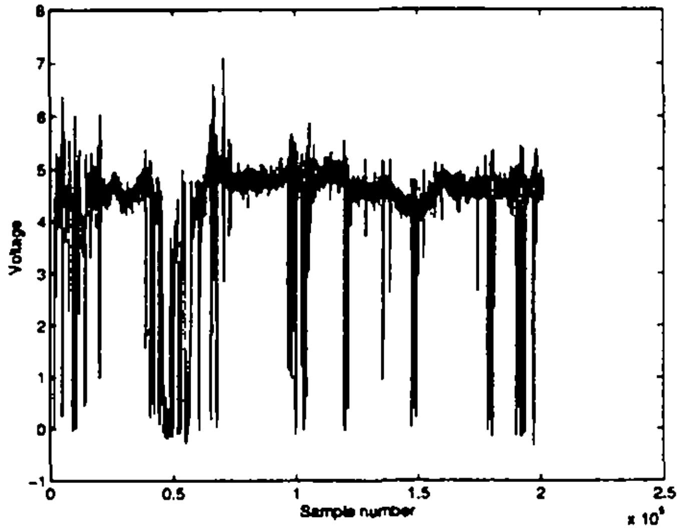

Figure 2.3: Plot of the received voltage levels for run 070906.

the satellite and the receiver is unobstructed. This voltage level is used as a reference. The value of λ was chosen for each run individually to represent each run's unobstructed signal mean received voltage level. A summary of the values of λ chosen and used in the analysis is in Table 2.2. In terms of the power received, the normalized received power level γ can be obtained by

$$\gamma = \frac{I^2}{\lambda^2}, \tag{2.1}$$

where I is the voltage level of the pilot data. For the purposes of analysis, a more useful form of γ is its value in dB: this is simply obtained by

$$\gamma_{\mathrm{dB}} = 10\log(\gamma). \tag{2.2}$$

Typical plots of γ_{dB} are shown Figures 2.5 to 2.9.

Any phase variations in the received signal are removed by the phase-locked loop (PLL), so the degradation in received signal power is due to obstructions in the line-of-sight path between the mobile receiver and the satellite transmitter. Fading is the complete or partial obstruction of the transmitted signal caused by the absorption and scattering of the incident direct signal by roadside trees or other obstacles in the path between the satellite and the vehicle [4]. The

Table 2.2: Values of λ for the various runs

RUN	CAT.	λ (V)
020201	I	3.0
070901	II	4.1
070903	III	4.5
070905	II	5.0
070906	III	4.9
070907	II	5.0
070912	III	5.0
070914	III	5.0
071016	II	5.5
071017	II	6.0
072405	II	4.0
072406	II	4.8
072407	II	5.0
072408	II	5.5
072409	II	5.0
072410	II	5.2
072411	II	5.5
072412	II	6.0

fading in the signal power level is apparent in Figures 2.5 through 2.9. When the received signal power level drops below a predetermined threshold level η, a *fade event* $\mathbf{r}$ occurs. Similarly, whenever the received signal power level is above the predetermined threshold, a *non-fade event* $\mathbf{R}$ occurs. A good example of fading is seen in Figure 2.4. In this Figure the received pilot power is plotted as a function of time. As is shown, each fade event is followed by a non-fade event, followed by another fade event and so on. In Figure 2.4 there are four fade events identified by $\mathbf{r}_{i-2}, \mathbf{r}_{i-1}, \mathbf{r}_i$, and $\mathbf{r}_{i+1}$. The subscript i simply identifies the fade events. Similarly, there are three non-fade events identified by $\mathbf{R}_{i-1}, \mathbf{R}_i$, and $\mathbf{R}_{i+1}$.

Let n_i be a random variable representing the duration of non-fade event $\mathbf{R}_i$. Once a non-fade event occurs, there is a probability that the duration of the event is at least n samples. The probability of a non-fade event lasting at least n samples is simply $\Pr(n_i > n)$, the complementary cumulative distribution of n_i. Similarly, f_j is a random variable that represents the duration of fade event $\mathbf{r}_j$. The probability of a fade event lasting at least n samples is $\Pr(f_j > n)$, the

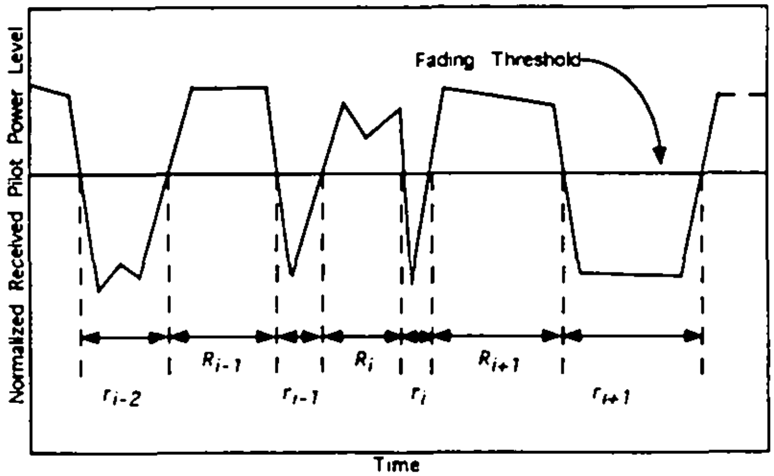

Figure 2.4: Fade and non-fade events.

complementary cumulative distribution of f_j. For the five sample runs to be used in this thesis, the fade and non-fade durations are given in figures 2.10 through 2.19.

The average fade and non-fade durations are obtained by taking the arithmetic average. There are N_{nf} non-fade events and N_f fade events. The average non-fade duration, $\overline{n}$, is given by

$$\overline{n} = \frac{\sum_{i=1}^{N_{nf}} n_i}{N_{nf}}. \tag{2.3}$$

The average fade duration $\overline{f}$ is given by

$$\overline{f} = \frac{\sum_{i=1}^{N_f} n_i}{N_f}. \tag{2.4}$$

The average fade and non-fade durations are given in Table 2.4 for $\eta = -6dB$ and $\eta = -10dB$. Note that in the table, by lowering the threshold level, the average fade duration decreases. Also, in many cases, the average non-fade duration decreases. This is due to the larger number of level crossings that occur at the lower threshold. There are more samples above the threshold, but due to the increased threshold crossings, there are also more non-fade duration regions. This results in a lower average non-fade duration, even though the total time of non-fade events is larger.

The duration of a fade or non-fade event can be measured by samples, distance or time. Let R_s be the sample rate and N_s be the number of samples corresponding to a particular fade or non-fade event (i.e. the duration of the event is N_s samples). It is sometimes useful to think of the duration of an event in terms of a physical distance since the receiver is mobile and fading is a spatial phenomenon. If the receiver travels at a constant velocity v, then each sample corresponds to a distance of

$$d_s = \frac{v}{R_s} \tag{2.5}$$

so that the duration of the event measured in distance is

$$D = N_s d_s. \tag{2.6}$$

Similarly, the duration of the event measured in time is

$$T = \frac{N_s}{R_s}. \tag{2.7}$$

The conversion of the sampled data into a bit stream is straightforward. Assuming a bit rate of R_b, the number of bits for each sample is related to the number of samples N_s by

$$N_b = \frac{R_b}{N_s}. \tag{2.8}$$

The bit rate herein is assumed to be 10 kbps.

2.5 Generation of Synthetic Error Sequences

For the case of digital data transmission over the channel, it is desirable to be able to model the errors that occur in order to characterize the channel. In the sampled data that was received, only a pilot tone signal level was recorded. Since the pilot tone contained no data, a synthetic error sequence was generated from the pilot tone information. The error sequences were based on the observation that, due to shadowing typical of the AMT system, the received signal is either high enough to permit error free communication or so low that signal acquisition is lost [12].

Table 2.3: Average Fade and Non-fade Duration Sample Lengths for Thresholds of -6dB and -10dB.

RUN	CAT.	$\overline{f}, \eta = -6dB$	$\overline{n}, \eta = -6dB$	$\overline{f}, \eta = -10dB$	$\overline{n}, \eta = -10dB$
070901	II	29	1857	12	1500
070903	III	132	257	85	198
070905	II	112	166	75	147
070906	III	81	690	55	597
070907	II	59	657	41	625
070912	III	170	246	107	187
070914	III	93	83	108	125
071016	II	59	1269	43	1132
071017	II	47	1075	27	850
072405	II	29	184	30	265
072406	II	36	687	25	649
072407	II	162	413	117	322
072408	II	51	899	24	613
072409	II	29	2260	15	1743
072410	II	111	133	72	106
072411	II	41	2976	24	2318
072412	II	92	416	58	312

In the work done by Humpherys [13], the probability density functions of the received signals for the various runs show that the majority of the received signal levels are concentrated around two regions. One of the regions is the mean received voltage level λ, which corresponds to error free communication. The other level is near zero, which is down in the noise and corresponds losing signal acquisition. It is because of these concentrations that the above assumption is made that the signal level is high enough to permit error free communication or so low that signal acquisition is lost. These assumptions are necessary in order to produce a synthetic error sequence. The ideal situation would be to have a known digital sequence that could be compared to the received sequence. In this case, the actual errors that occurred across the channel would be known directly, and the generation of a synthetic error sequence would be unnecessary. However, in this case, the only available information is the received signal levels, and therefore the best alternative is to generate a sequence that represents the errors that occur on this type of channel.

With the above assumptions, the fade events are associated with *error producing states* and the non-fade events are associated with *error free states*. The error and error free states are used to produce the synthetic error sequence $\mathbf{e}$. A synthetic error sequence of L bits is represented by

$$\mathbf{e} = e_0 e_1 e_2 \cdots e_{L-1} \tag{2.9}$$

where

$$e_i = \begin{cases} 0 & \text{if } \gamma_i \geq \eta \\ b & \text{if } \gamma_i < \eta \end{cases} \tag{2.10}$$

where $b \in \{0, 1\}$ is a binary random variable with

$$\Pr\{b = 0\} = \Pr\{b = 1\} = 1/2. \tag{2.11}$$

A synthetic error sequence $\mathbf{e}$ was produced for each data run.

2.6 The Error Gap Distribution and Block Error Probabilities

An error gap is defined as a string of consecutive zeros between two ones in the error sequence. The length U of the error gap is equal to one plus the number of zeros between the two ones. Since U is a random variable it has a distribution

$$u(n) = \Pr\{U \geq n\} \tag{2.12}$$

which gives the probability that there are at least n error free bits between errors.

An alternate characterization of the error sequence is the block error probability $P(m, n)$ which is the probability that m errors occur in a block of n consecutive error bits. In any particular error sequence, the total number of n-length blocks varies directly with the total length of the error sequence L in bits. From block to block in the error sequence, the number of errors in each block will vary as the error and error free producing states are affected by the received signal power levels. Also, the block being examined may start at any point in the error sequence. Thus, the block will have a varying number of errors depending where the block starts. Because of this, the average value of $P(m, n)$, denoted $\overline{P}(m, n)$, is computed for each synthetic error sequence.

The number of errors for any block is obtained by counting the number of errors that occur in that block. Let $\mathbf{e}_i$ be a block of n consecutive bits in $\mathbf{e}$ starting at bit e_i:

$$\mathbf{e}_i = [e_i, e_{i+1}, e_{i+2}, ..., e_{i+n-1}] \tag{2.13}$$

where $i = 0, 1, \cdots, L-n+1$. The number of errors in block $\mathbf{e}_i$ is given by the Hamming weight of the block

$$w_H(\mathbf{e}_i) = \sum_{j=i}^{i+n-1} e_j. \tag{2.14}$$

For each error sequence $\mathbf{e}$, the average block error probability $\overline{P}(m,n)$ is estimated by dividing the number of blocks $\mathbf{e}_i$ which contain exactly m errors by the total number of blocks B:

$$\overline{P}(m,n) = \frac{|\{i|w_H(\mathbf{e}_i) = m\}|}{B} \tag{2.15}$$

where $m = 0, 1, 2, \cdots, n$.

There are several different models that have been proposed that use the error gap distribution and the block error probabilities to model the error statistics of various channels. The following chapters examine four of these models and assesses their applicability to the ACTS data. The first model to be presented is the Gilbert model.

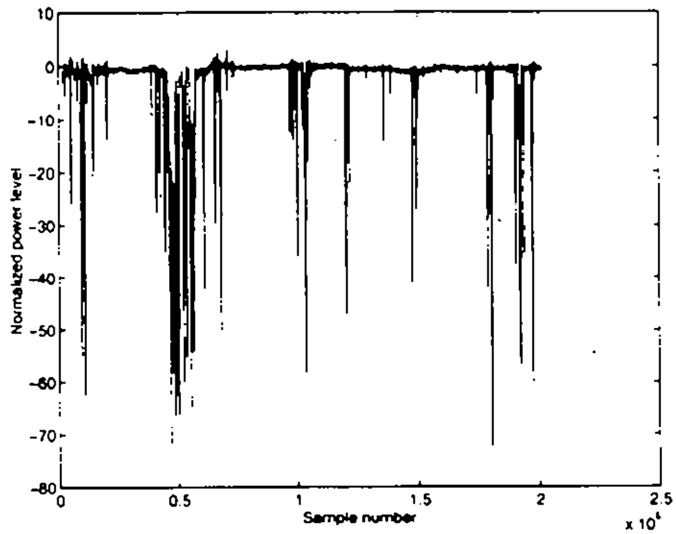

Figure 2.5: Plot of normalized received power level for run 070906.

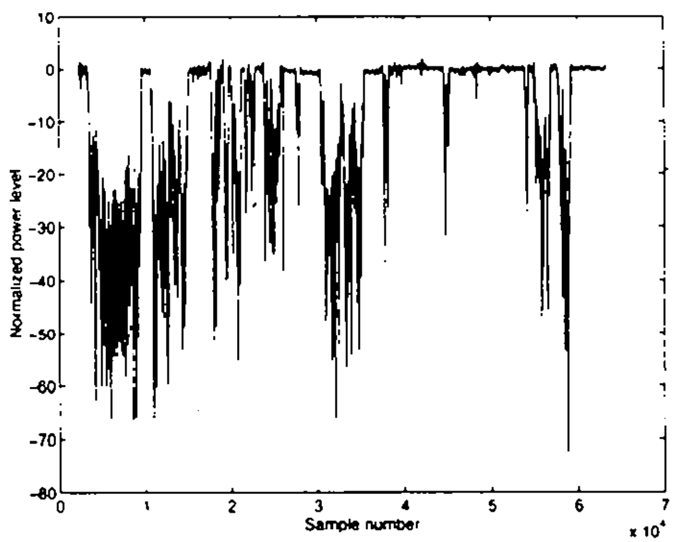

Figure 2.6: Plot of normalized received power level for run 070912.

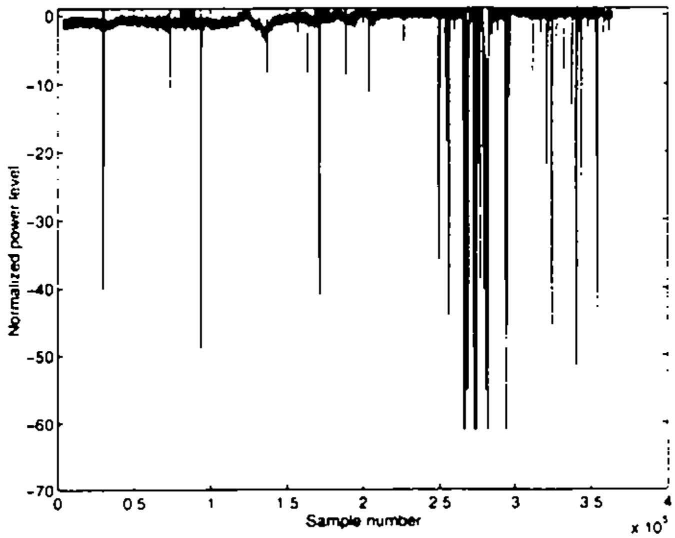

Figure 2.7: Plot of normalized received power level for run 071016.

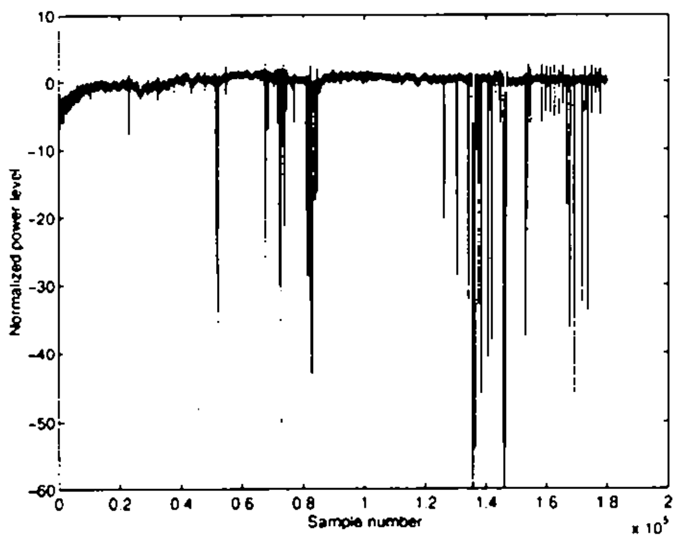

Figure 2.8: Plot of normalized received power level for run 072406.

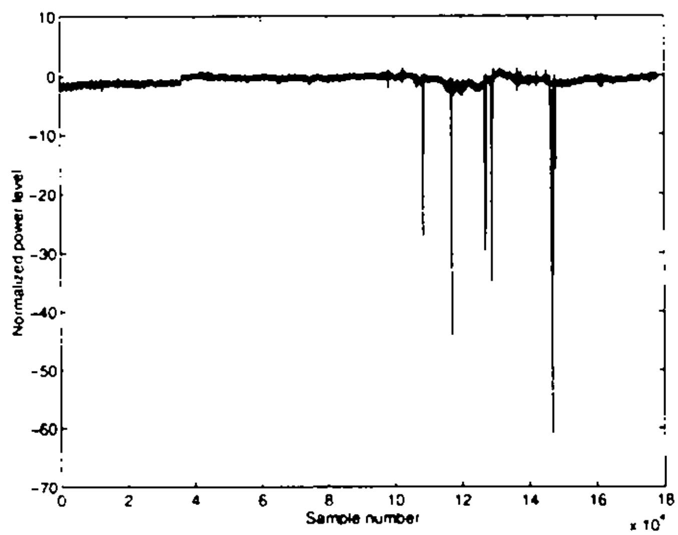

Figure 2.9: Plot of normalized received power level for run 072411.

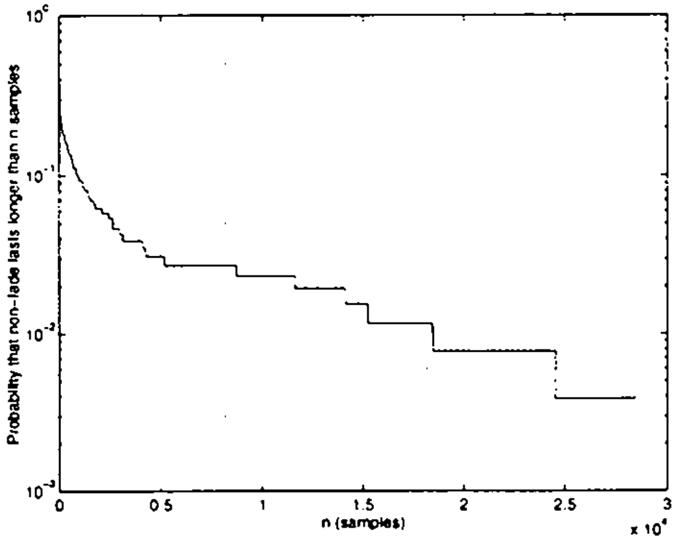

Figure 2.10: Plot of the non-fade duration for run 070906 for $\eta = -6dB$

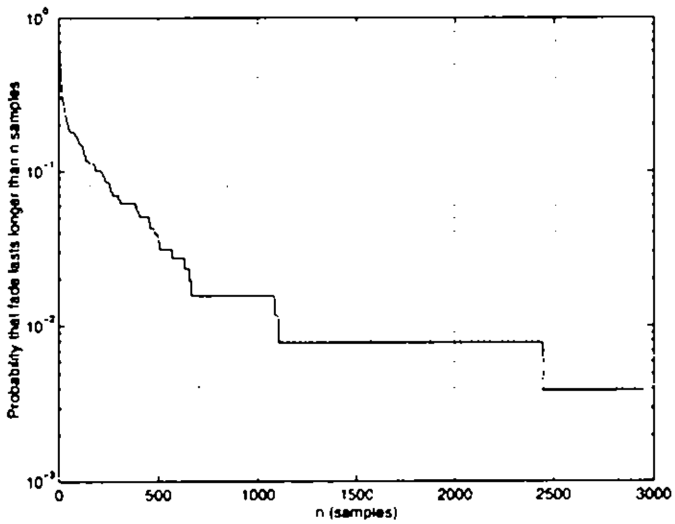

Figure 2.11: Plot of the fade duration for run 070906 for $\eta = -6$ dB

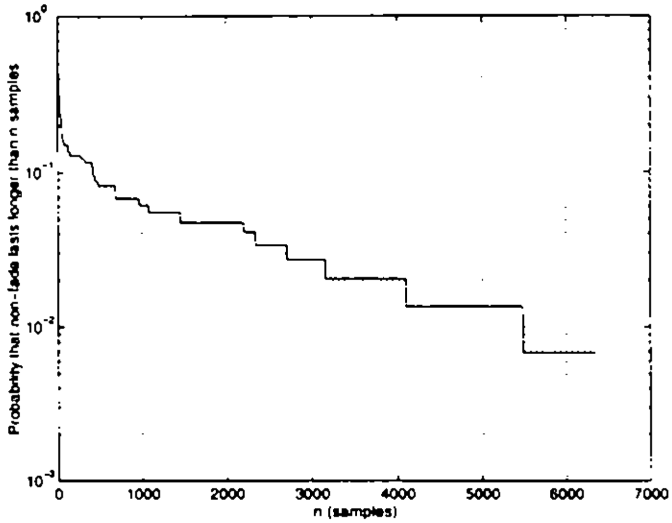

Figure 2.12: Plot of the non-fade duration for run 0709012 for η = −6 dB

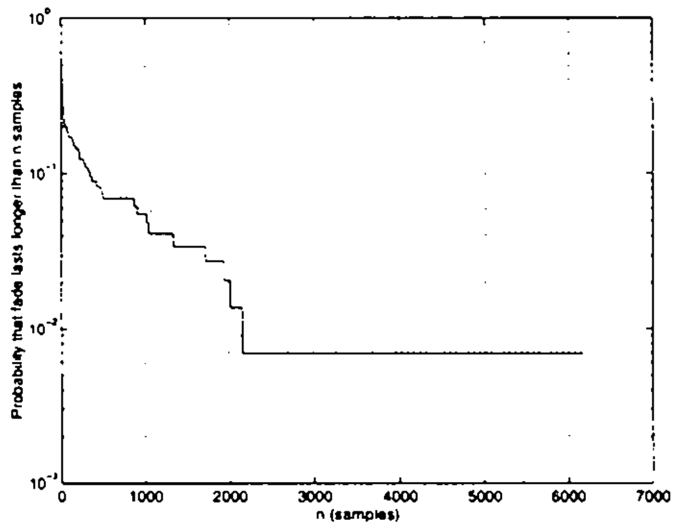

Figure 2.13: Plot of the fade duration for run 070912 for η = −6 dB

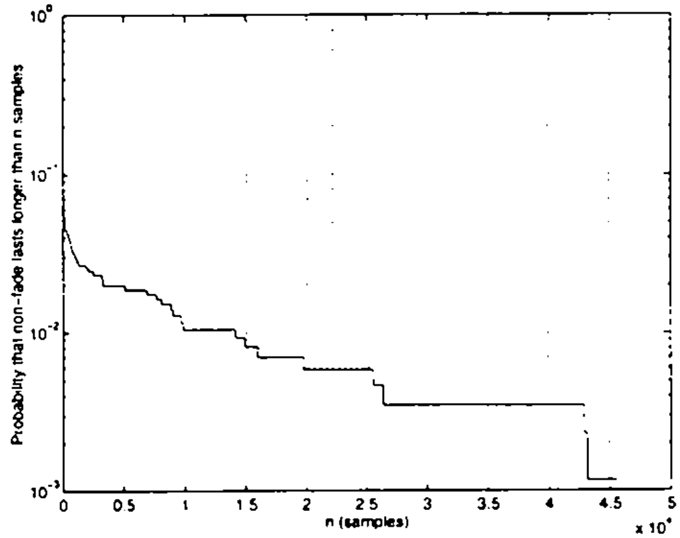

Figure 2.14: Plot of the non-fade duration for run 071016 for $\eta = -6$ dB

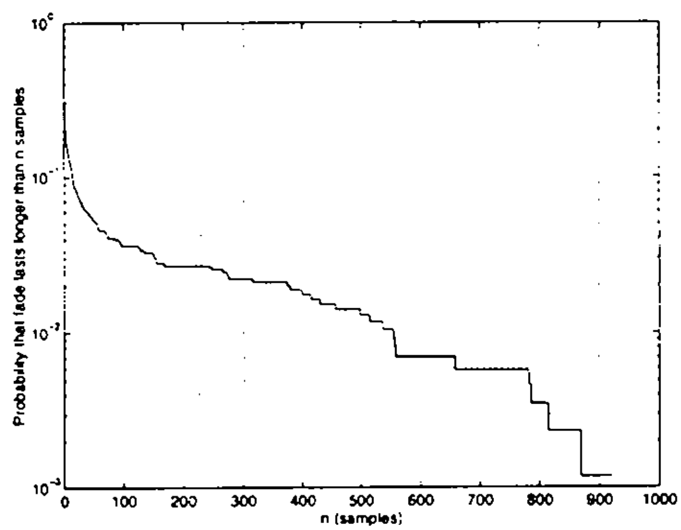

Figure 2.15: Plot of the fade duration for run 071016 for $\eta = -6$ dB

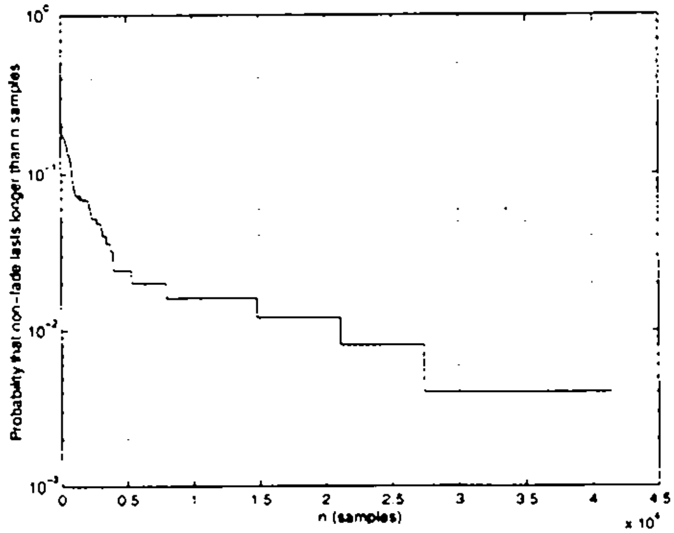

Figure 2.16: Plot of the non-fade duration for run 072406 for $\eta = -6$ dB

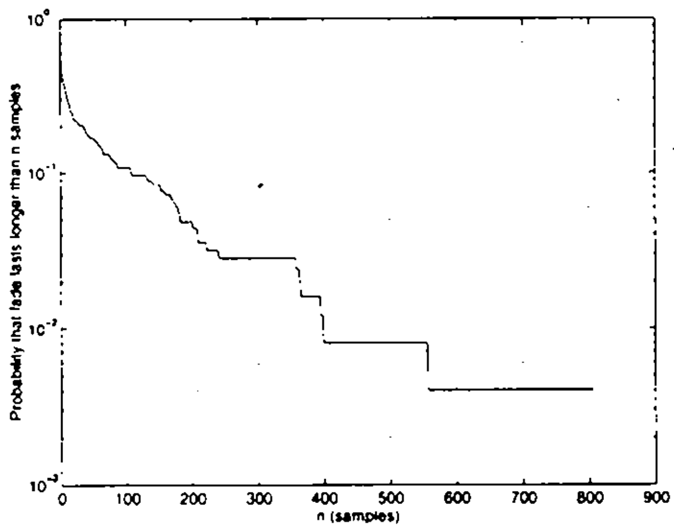

Figure 2.17: Plot of the fade duration for run 072406 for $\eta = -6$ dB

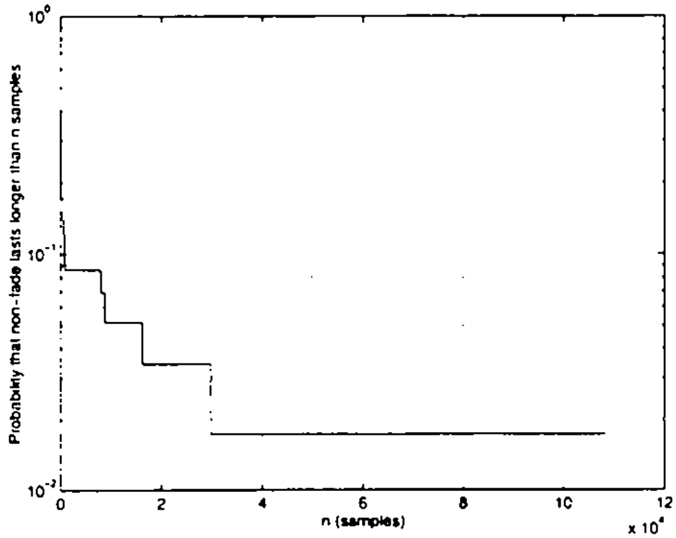

Figure 2.18: Plot of the non-fade duration for run 072411 for $\eta = -6$ dB

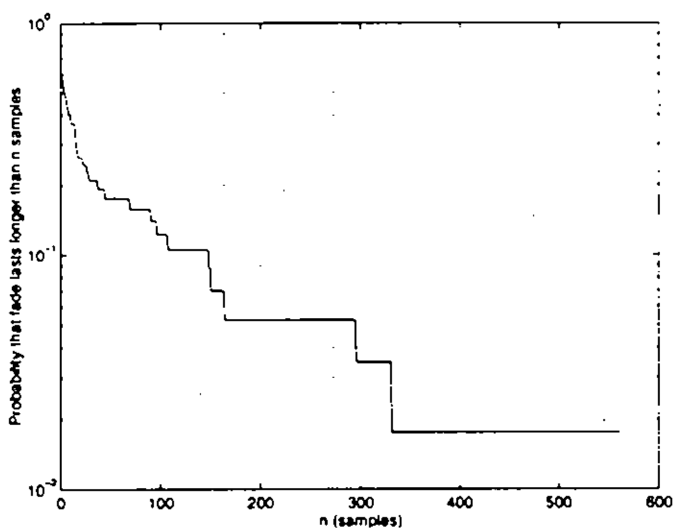

Figure 2.19: Plot of the fade duration for run 072411 for $\eta = -6$ dB

Chapter 3

THE GILBERT MODEL

The Gilbert model is an important starting point because it is the simplest model and one on which more complicated models are based. The Gilbert model is a two-state Markov model where each state represents a different binary symmetric channel (BSC). One channel is a clear channel, and the other is a "noisy" channel.

3.1 The Model

The Gilbert model [5] shown in Figure 3.1 is a Markov chain containing two states, a "good state" S_1, and a "bad state" S_2. State S_1 represents a binary symmetric channel where the probability of a bit error is zero, hence the noise digit e_i is always a zero. This is equivalent to a clear channel where no errors occur. State S_2 represents a BSC where the probability of a bit error is $1 - h$, which is equivalent to the noise digit e_i being zero with probability h.

The transition probabilities between states in this Markov chain are represented by a state transition matrix P, where

$$P_{\mathrm{ij}} = \Pr\{\text{next bit producing state} = S_{\mathrm{j}} | \text{current bit producing state} = S_{\mathrm{i}}\}. \tag{3.1}$$

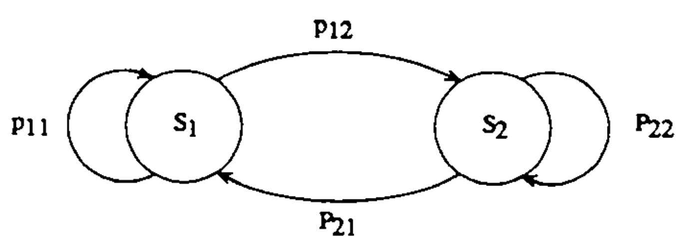

Figure 3.1: The Gilbert Model.

3.2 Error Gap Distribution and Block Error Probabilities

Gilbert derived the relationship between the transition probabilities, h, and the error gap distribution. Following the development in [5], the error gap distribution and these relations are derived below.

Let f_n denote the conditional probability that, starting in state S_2, the first return to S_2 will occur after n transitions(or bits):

$$f_n = P(S_1^{n-1}S_2|S_2). \tag{3.2}$$

Then $f_1 = P_{22}$ and $f_n = P_{21}{P_{11}}^{n-2}P_{12}$ for $n \geq 2$. Making these probabilities the coefficients of a generating function $F(t)$ of recurrence time probabilities,

$$F(t) = \sum f_n t^n = P_{22}t + \frac{P_{21}P_{12}t^2}{1 - P_{11}t}. \tag{3.3}$$

The probability that the mth return to S_2 happens at bit n has the generating function

$$\sum_{n=1}^{\infty} f_n^{(m)}t^n = [F(t)]^m. \tag{3.4}$$

The probability of no return to S_2 in k bits is $P_{21}{P_{11}}^{k-1}$. Then the probability $s(n,m)$ of exactly m returns to S_2 in n bits (but not necessarily a return on bit n) is

$$s(n,m) = f_n^{(m)} + \sum_{n=1}^{n-m} f_{n-k}^{(m)}P_{21}{P_{11}}^{k-1}. \tag{3.5}$$

The corresponding generating function is

$$\sum_{n=1}^{\infty} s(n,m)t^n = \left(1 + \frac{P_{21}t}{1 - P_{11}t}\right)[F(t)]^m. \tag{3.6}$$

Now, starting from a one in state S_2, the next one must occur at a return to state S_2. The probability that the next one occurs at the mth return to S_2 and at bit n is

$$h^{m-1}(1-h)f_n^{(m)}. \tag{3.7}$$

Then, recurrence time probabilities for ones are

$$\begin{aligned} v(n-1) &= P(0^{n-1}|1) \\ &= \sum_{m=1}^{\infty} h^{m-1}(1-h)f_n^{(m)} \end{aligned}$$

where $v(n) = P(0^n1|1)$. From (3.4), the generating function is $V(t) = \sum v(n)t^n$. Similarly, the probability $u(n)$ that no one appears in the next n bits is

$$u(n) = \sum_m s(n,m)h^m, \tag{3.8}$$

which has the generating function

$$U(t) = \frac{1 + (P_{21} - P_{11})t}{(1 - P_{11}t)[1 - hF(t)]}. \tag{3.9}$$

From (3.3),

$$U(t) = \frac{1 + (P_{21} - P_{11})t}{D(t)}, \tag{3.10}$$

where $D(t) = 1 - (P_{11} + hP_{22})t - h(P_{21} - P_{11})t^2$. Factoring the quadratic $D(t)$,

$$D(t) = (1 - Jt)(1 - Lt),$$

where

$$\begin{aligned} J &= \frac{P_{11} + hP_{22} + \sqrt{(P_{11} + hP_{22})^2 + 4h(P_{21} - P_{11})}}{2} \\ L &= \frac{P_{11} + hP_{22} - \sqrt{(P_{11} + hP_{22})^2 + 4h(P_{21} - P_{11})}}{2}. \end{aligned}$$

Now, (3.10) becomes

$$U(t) = \frac{1 + (P_{21} - P_{11})t}{J - L}\left(\frac{J}{1 - Jt} - \frac{L}{1 - Lt}\right).$$

The coefficient of t^n for the power series $U(t)$ is

$$u(n) = \frac{(J + P_{21} - P_{11})J^n - (L + P_{21} - P_{11})L^n}{J - L}. \tag{3.11}$$

Letting $A = (J + P_{21} - P_{11})/(J - L)$, (3.11) can be rewritten as

$$u(n) = AJ^n + (1 - A)L^n. \tag{3.12}$$

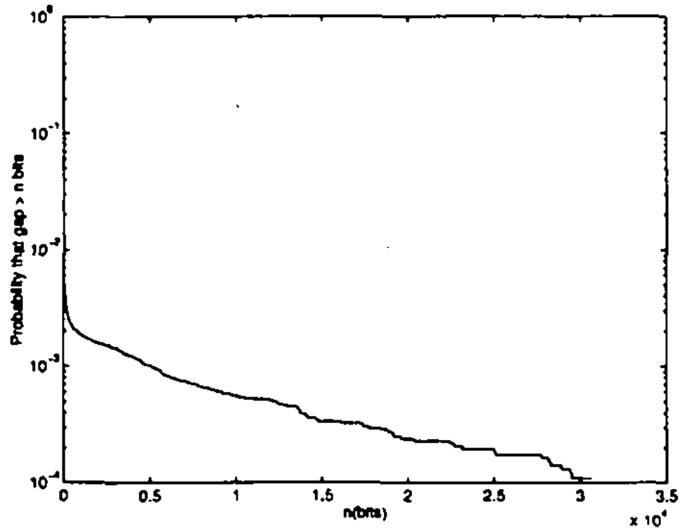

Figure 3.2: Typical error gap distribution curve.

Equation (3.12) is the error gap distribution for the Gilbert model. A typical error gap distribution curve shown in Figure 3.2 shows that there are two dominating slopes. One is a steeper slope for smaller values of n and the other is a much less steep slope for the larger values of n.

From the error gap distribution plot and equation (3.12), parameters $A, J,$ and$L$ are found by first fitting the curve to the larger values of n by choosing A and J. Once these are chosen, then the fit for smaller values of n is obtained by selecting L. These values are then used to derive the value of h and the transition probabilities P_{12} and P_{21}, given by

$$h = \frac{LJ}{J - A(J - L)} \tag{3.13}$$

$$P_{12} = \frac{(1 - L)(1 - J)}{1 - h} \tag{3.14}$$

and

$$P_{21} = A(J - L) + (1 - J)(\frac{L - h}{1 - h}) \tag{3.15}$$

Using these parameters, the block error probabilities are found using a set of recurrence relations [6]. The general form of $P(m, n)$ for a two-state BSC Markov model is given by

$$P(m,n) = \frac{P_{21}}{P_{21}+P_{12}} S_1(m,n) + \frac{P_{12}}{P_{21}+P_{12}} S_2(m,n) \tag{3.16}$$

where $S_1(m,n)$ and $S_2(m,n)$ are given in recurrent form by

$$S_1(m,n) = S_1(m,n-1)P_{11} + S_2(m,n-1)P_{12} \tag{3.17}$$

and

$$\begin{aligned} S_2(m,n) = {} & S_2(m,n-1)P_{22}h + S_1(m,n-1)P_{21}h \\ & + S_2(m-1,n-1)P_{22}(1-h) + S_1(m-1,n-1)P_{21}(1-h). \end{aligned} \tag{3.18}$$

The initial conditions used in computing the above values for Gilbert's model are

$$\begin{aligned} S_1(0,1) &= 1 \\ S_1(1,1) &= 0 \\ S_2(0,1) &= h \\ S_2(1,1) &= 1-h \end{aligned}$$

and

$$S_1(m,n) = S_2(m,n) = 0, \text{when } m < 0 \text{ or } m > n.$$

Using these initial conditions and a set block length n, the error gap distribution and block error statistics for the data and the model are ready to be compared. For the Gilbert and subsequent models, a block of length $n = 64$ is used.

3.3 Comparison of Gilbert Model and the Data

There are two approaches in determining the model parameters which best match the data.

1. Determine the values of A, J, and L which provide the closest fit for the error gap distribution. From these parameters, block error probabilities can be derived.

Table 3.1: Gilbert Model Parameters Chosen to Match the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999983	0.676855	0.002977	0.678	1.75×10^{-5}	9.62×10^{-4}
070903	III	0.999930	0.588964	0.000498	0.589	6.97×10^{-5}	2.05×10^{-4}
070905	II	0.999906	0.622168	0.000450	0.622	9.31×10^{-5}	1.70×10^{-4}
070906	III	0.999965	0.595634	0.000684	0.596	3.47×10^{-5}	2.77×10^{-4}
070907	II	0.999947	0.621582	0.001244	0.622	5.26×10^{-5}	4.71×10^{-4}
070912	III	0.999901	0.5617968	0.000367	0.562	9.87×10^{-5}	1.61×10^{-4}
070914	III	0.999735	0.611230	0.000521	0.611	2.65×10^{-4}	2.02×10^{-4}
071016	II	0.999991	0.674779	0.000668	0.675	8.10×10^{-6}	2.18×10^{-4}
071017	II	0.999993	0.750000	0.000539	0.750	6.52×10^{-6}	1.35×10^{-4}
072405	II	0.999914	0.984838	0.001494	0.985	8.62×10^{-5}	2.24×10^{-5}
072406	II	0.999957	0.628417	0.001708	0.629	4.23×10^{-5}	6.35×10^{-4}
072407	II	0.999925	0.567187	0.000466	0.567	7.44×10^{-5}	2.02×10^{-4}
072408	II	0.999975	0.628417	0.001135	0.629	2.41×10^{-5}	4.22×10^{-4}
072409	II	0.999971	0.628710	0.003114	0.629	2.87×10^{-5}	1.16×10^{-3}
072410	II	0.999856	0.603027	0.000505	0.603	1.43×10^{-4}	2.00×10^{-4}
072411	II	0.999952	0.621386	0.002044	0.622	4.73×10^{-5}	7.74×10^{-4}
072412	II	0.999930	0.597949	0.000715	0.598	7.00×10^{-5}	2.88×10^{-4}

2. Determine the values of the transition probabilities and h which provide the closest fit for the block error probabilities. Then, the error gap distribution parameters can be derived.

The error gap distributions and the model curve fits chosen to best fit the error gap distributions are shown in Figures 3.3, 3.5, 3.7, 3.9, and 3.11 for runs 070906, 070912, 071016, 072406, and 072411 respectively. Using mean-square error as a measure of closeness of fit, Table 3.1 shows the error gap distribution parameters and the associated derived transition probabilities along with h.

It can be seen from the curves that the $u(n)$ for the model is a good approximation to the measured data for the larger values of n, but the smaller values of n do not fit as well. Hence this model applies more toward modeling the probabilities of the larger gaps in the error gap distribution than the smaller ones.

Checking the derived block error statistics for this model, a block of size $n = 64$ is used. Using the parameters that determine $u(n)$, $P(m,n)$ is shown in

Table 3.2: Gilbert Model Parameters Chosen to Match the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999968	0.522116	3.50×10^{-3}	0.523	3.17×10^{-5}	1.67×10^{-3}
070903	III	0.999750	0.4987624	9.78×10^{-4}	0.499	2.50×10^{-4}	4.90×10^{-4}
070905	II	0.999580	0.493197	1.20×10^{-3}	0.493	4.2×10^{-4}	6.06×10^{-4}
070906	III	0.999904	0.4975809	1.62×10^{-3}	.497	9.55×10^{-5}	8.13×10^{-4}
070907	II	0.999892	0.491313	2.34×10^{-3}	0.490	1.08×10^{-4}	1.19×10^{-3}
070912	III	0.999773	0.501307	7.01×10^{-4}	0.501	2.27×10^{-4}	3.49×10^{-4}
070914	III	0.999428	0.501717	1.03×10^{-3}	0.502	5.72×10^{-4}	5.10×10^{-4}
071016	II	0.999962	0.520923	1.39×10^{-3}	0.520	3.80×10^{-5}	6.66×10^{-4}
071017	II	0.999899	0.241149	2.11×10^{-3}	0.521	1.01×10^{-4}	1.60×10^{-3}
072405	II	0.999693	0..496921	3.94×10^{-3}	0.497	3.08×10^{-4}	1.98×10^{-3}
072406	II	0.999926	0.501599	2.81×10^{-3}	0.502	7.32×10^{-5}	1.40×10^{-3}
072407	II	0.999845	0.4929315	7.62×10^{-4}	0.493	1.55×10^{-4}	3.86×10^{-4}
072408	II	0.999944	0.502804	1.97×10^{-3}	0.503	5.60×10^{-5}	9.79×10^{-4}
072409	II	0.999972	0.496521	4.49×10^{-3}	0.497	2.81×10^{-5}	2.26×10^{-3}
072410	II	0.999554	0.495626	9.99×10^{-4}	0.495	4.46×10^{-4}	5.03×10^{-4}
072411	II	0.999983	0.522043	2.09×10^{-3}	0.522	1.74×10^{-5}	9.96×10^{-4}
072412	II	0.999856	0.502513	1.34×10^{-3}	0.502	1.43×10^{-4}	6.65×10^{-4}

Figures 3.4, 3.6, 3.8, 3.10, and 3.12.

It is readily apparent from the comparison in these figures that the Gilbert model for the block error probabilities is a conservative estimate of the block error probabilities. For smaller values of m the curve fits are fair. But as m increases, the "hump" in the data occurs later than the model predicts. This implies that the Gilbert model has some limitations in its ability to accurately model the block error probabilities for the channel given the error gap distribution.

To determine whether the Gilbert model can model $\overline{P}(m,n)$ with any accuracy, an exhaustive search of possible values of h, P_{12}, and P_{21} using mean-square error as a measure of accuracy to $\overline{P}(m,n)$ was performed. The values in Table 3.2 were obtained along with the associated error gap distribution parameters that would produce these corresponding values of $P(m,n)$. A comparison of $\overline{P}(m,n)$ and $P(m,n)$ derived from these parameters are shown in Figures 3.13, 3.15, 3.17, 3.19, and 3.21. The Figures show that the models, when given appropriate parameters, come much closer to matching $\overline{P}(m,n)$than before. The models vary in accuracy depending on the run, but overall, the model gives a very good estimate of the block error probabilities. The resulting effect on the error gap distribution by altering the state transition probabilities and h to match the block error data is shown in Figures 3.14, 3.16, 3.18, 3.20, and 3.22. In these Figures it is seen that by finding parameters that more closely approximate $\overline{P}(m,n)$ results in the model's curves of the error gap distribution values departing slightly from the data curves.

From this it is concluded that by using the Gilbert model, $u(n)$ for large n can approximate the error gap distribution. Also, $P(m,n)$ can model the block error probabilities. But both curves cannot be modeled simultaneously. An extension of the Gilbert model is considered next which provides some more flexibility in modeling the block error probabilities given the error gap distribution.

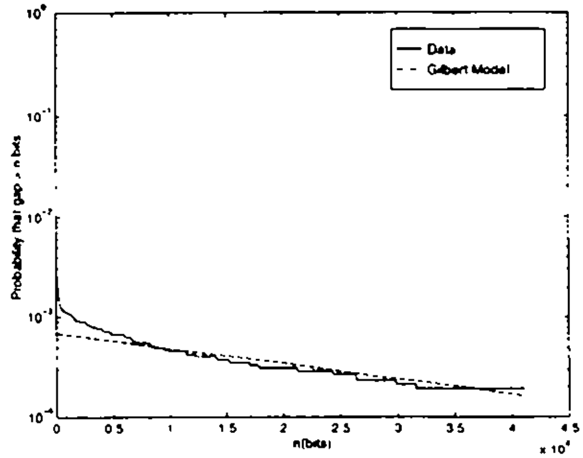

Figure 3.3: Error Gap Distribution Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 070906.

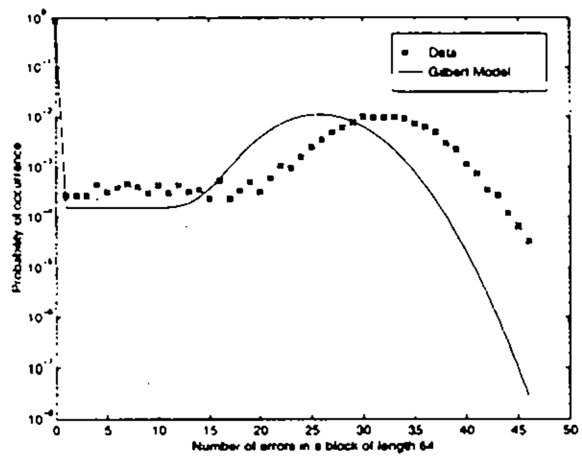

Figure 3.4: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 070906.

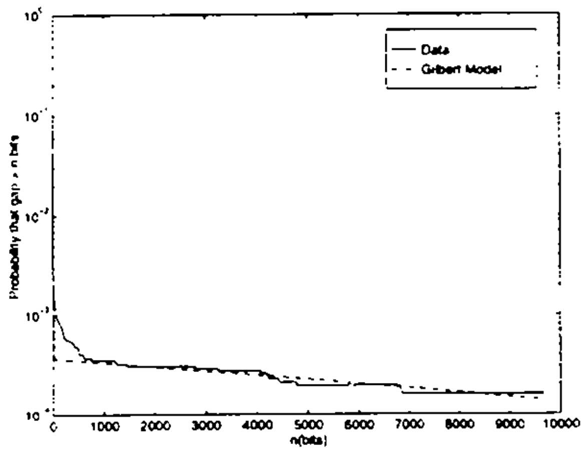

Figure 3.5: Error Gap Distribution Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 070912.

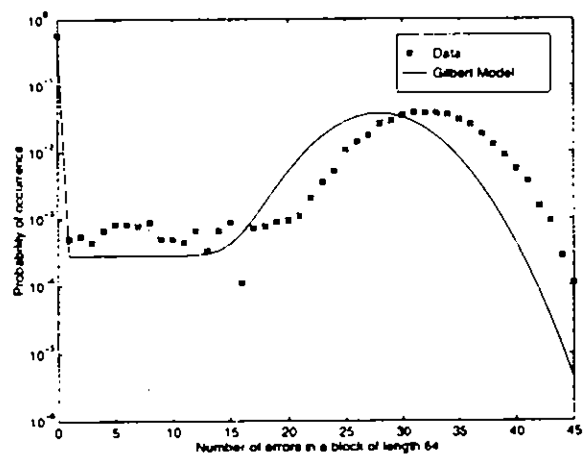

Figure 3.6: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 070912.

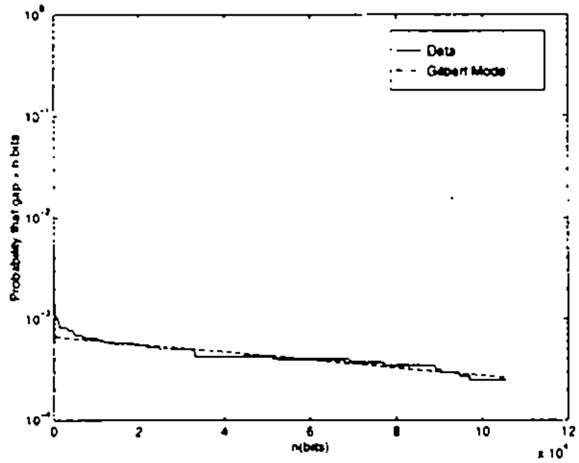

Figure 3.7: Error Gap Distribution Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 071016.

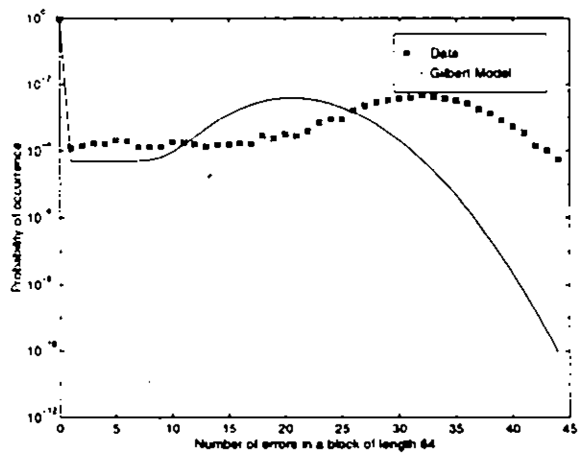

Figure 3.8: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 071016.

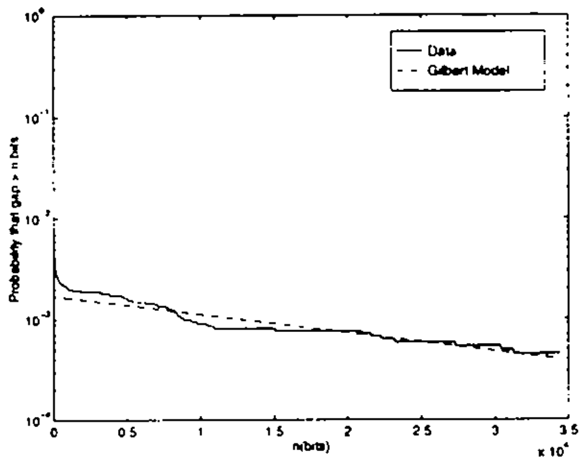

Figure 3.9: Error Gap Distribution Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 072406.

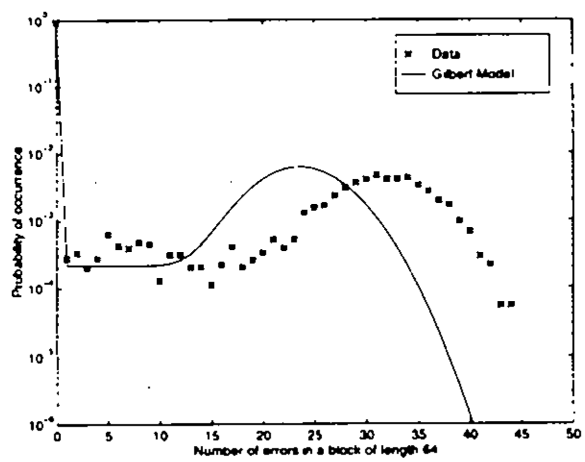

Figure 3.10: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 072406.

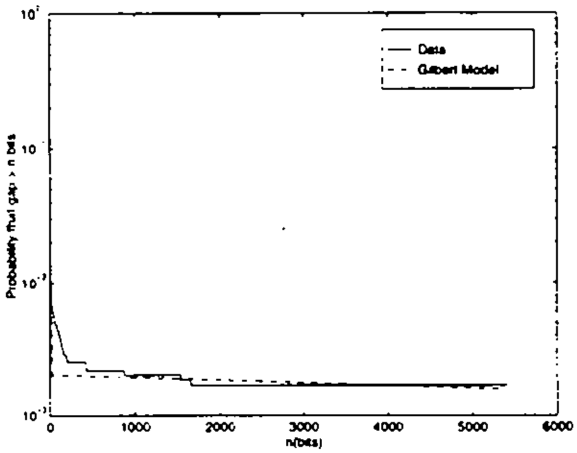

Figure 3.11: Error Gap Distribution Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 072411.

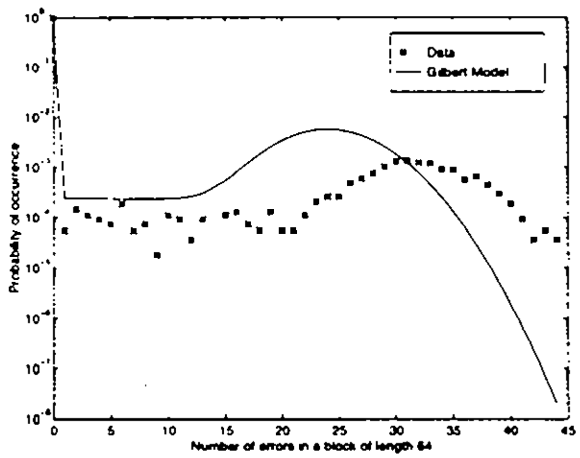

Figure 3.12: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Error Gap Distribution for Run 072411.

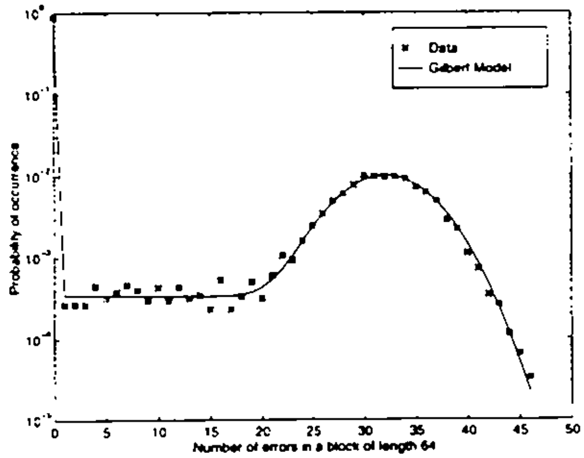

Figure 3.13: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Block Error Probabilities for Run 070906.

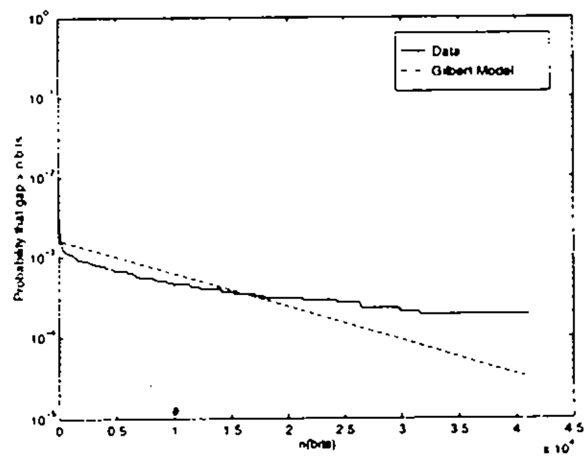

Figure 3.14: Error Gap Distribution Using the Gilbert model with Parameters Chosen to Match the Block Error Probabilities for Run 070906.

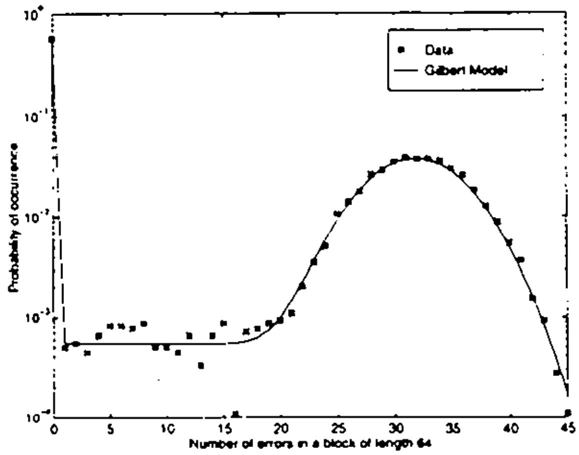

Figure 3.15: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Block Error Probabilities for Run 070912.

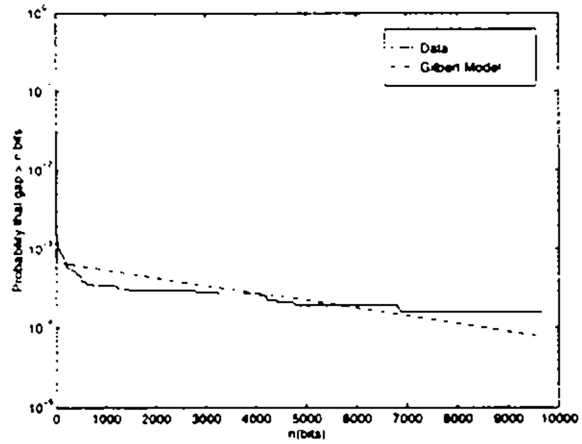

Figure 3.16: Error Gap Distribution Using the Gilbert model with Parameters Chosen to Match the Block Error Probabilities for Run 070912.

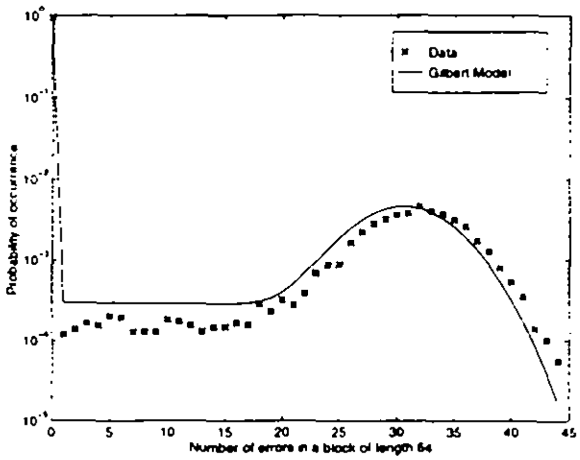

Figure 3.17: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Block Error Probabilities for Run 071016.

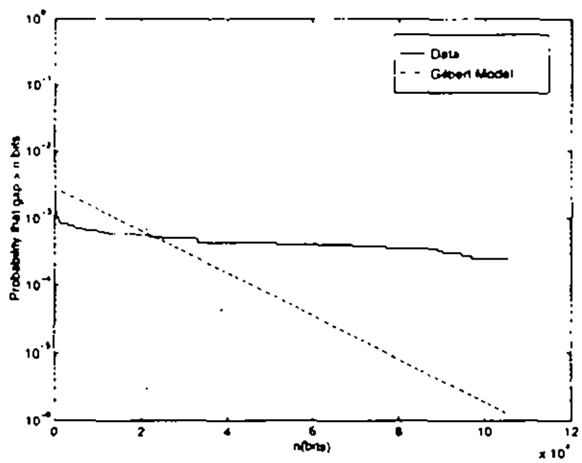

Figure 3.18: Error Gap Distribution Using the Gilbert model with Parameters Chosen to Match the Block Error Probabilities for Run 071016.

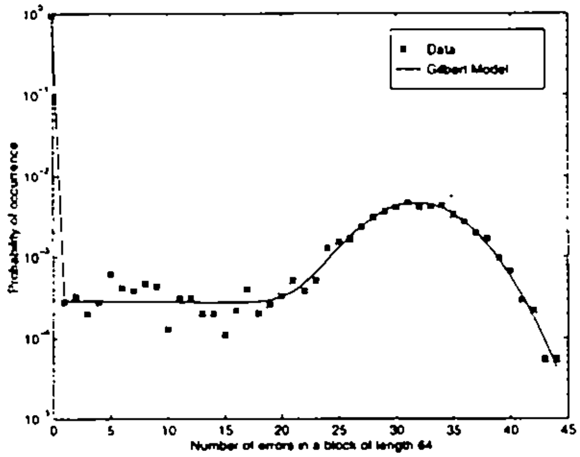

Figure 3.19: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Block Error Probabilities for Run 072406.

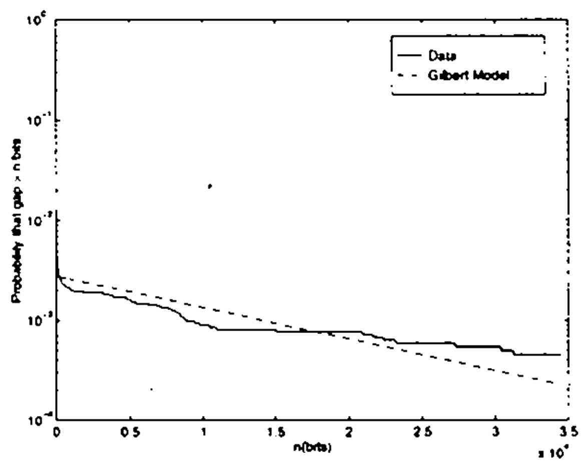

Figure 3.20: Error Gap Distribution Using the Gilbert model with Parameters Chosen to Match the Block Error Probabilities for Run 072406.

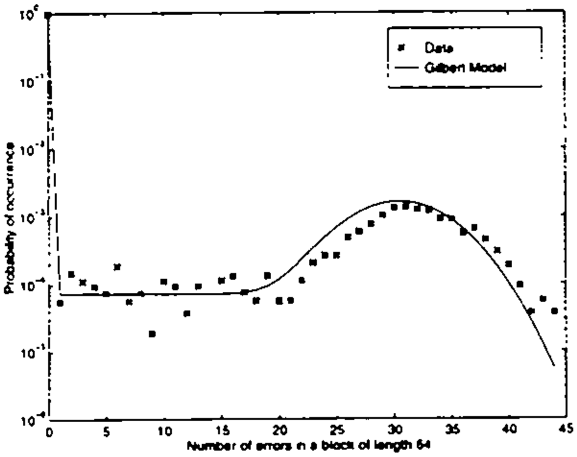

Figure 3.21: Block Error Probabilities Using the Gilbert Model with Parameters Chosen to Match the Block Error Probabilities for Run 072411.

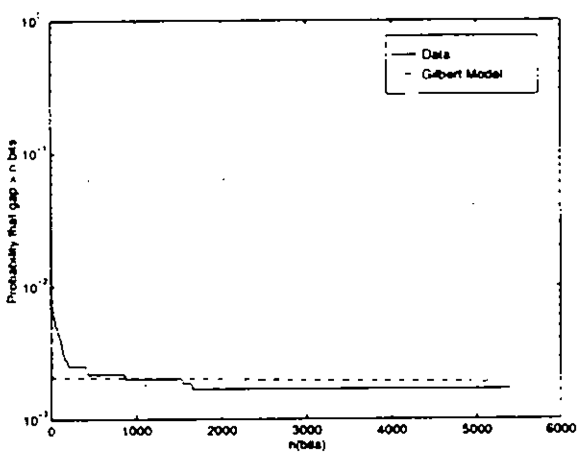

Figure 3.22: Error Gap Distribution Using the Gilbert model with Parameters Chosen to Match the Block Error Probabilities for Run 072411.

Chapter 4

THE ELLIOTT MODEL

The simplicity of the Gilbert model limits its ability to model simultaneously both the error gap distribution and the block error probabilities. Also, in modeling either one aspect or the other, it is only partially successful as seen in the previous chapter. The Elliott model, which is an extension of the Gilbert model, is a two state Markov model with each state representing a binary symmetric channel. One channel is a "noisy" channel, which has bursty errors, while the other is a random error channel.

4.1 The Model

Like the Gilbert Model, the Elliott model [6] shown in Figure 4.1 has two states, a "good state" S_1 and a "bad state" S_2. State S_1 is a BSC where the probability of the noise digit e_i being zero is k. State S_2 is a BSC where the probability of the noise digit e_i being zero is h. In this way, the model represents a "good" state where random errors occur infrequently, and a "bad" state where errors occur more frequently and have more of a bursty nature. The parameter k allows this model to have a little more flexibility in producing errors. The entries in the state transition matrix are given by (3.1).

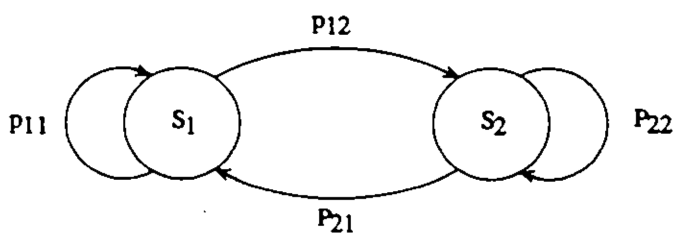

Figure 4.1: The Elliott Model.

4.2 The Error Gap Distribution and the Block Error Probabilities

Elliott has proposed a relationship between the transition probabilities and the block error probabilities that is based on Gilbert's approach. First, the error gap distribution is parameterized using (3.12) and the transition probabilities and h are derived using (3.13), (3.14), and (3.15). Next, in determining $P(m,n)$, the parameter k is incorporated in an attempt to make a better fit to $\overline{P}(m,n)$.

As mentioned above, the error gap distribution is modeled the same way as in the Gilbert model. While there will be some minor differences in the gap distribution due to the addition of the parameter k, it is assumed that the value of k is close to one, and will have only a minor effect on the distribution values. Its effect on the error gap distribution is therefore ignored. The values generated for $P(m,n)$, however, must include the effect of the parameter k. Note that Elliott's model reduces to Gilbert's model if $k = 1$.

The following block error probability equations are altered to reflect the inclusion of k. As in the Gilbert model,

$$P(m,n) = \frac{P_{21}}{P_{21}+P_{12}} S_1(m,n) + \frac{P_{12}}{P_{21}+P_{12}} S_2(m,n). \tag{4.1}$$

The recurrent forms of $S_1(m,n)$ and $S_2(m,n)$ are

$$\begin{aligned} S_1(m,n) = {} & S_1(m,n-1)P_{11}k + S_2(m,n-1)P_{12}k \\ & + S_1(m-1,n-1)P_{11}(1-k) + S_2(m-1,n-1)P_{12}(1-k) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} S_2(m,n) = {} & S_2(m,n-1)P_{22}h + S_1(m,n-1)P_{21}h \\ & + S_2(m-1,n-1)P_{22}(1-h) + S_1(m-1,n-1)P_{21}(1-h). \end{aligned} \tag{4.3}$$

The initial conditions are

$$\begin{aligned} S_1(0,1) &= k \\ S_1(1,1) &= 1-k \\ S_2(0,1) &= h \\ S_2(1,1) &= 1-h \\ S_1(m,n) &= S_2(m,n) = 0, \text{when } m < 0 \text{ or } m > n. \end{aligned}$$

Table 4.1: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	J	L	A
070901	II	0.999983	0.676855	0.002977
070903	III	0.999930	0.58896	0.000498
070905	II	0.999906	0.622168	0.000450
070906	III	0.999965	0.595634	0.000684
070907	II	0.999947	0.621580	0.001244
070912	III	0.999901	0.561796	0.000367
070914	III	0.999735	0.611230	0.000521
071016	II	0.999991	0.674779	0.000668
071017	II	0.999993	0.750000	0.000539
072405	II	0.999914	0.984838	0.001494
072406	II	0.999957	0.628417	0.001708
072407	II	0.999925	0.567187	0.000466
072408	II	0.999975	0.628417	0.001135
072409	II	0.999971	0.628710	0.003114
072410	II	0.999856	0.603027	0.000505
072411	II	0.999952	0.621386	0.002044
072412	II	0.999930	0.597949	0.000715

From these equations one can see that the effect of k appears in both (4.2) and in the initial conditions.

4.3 Comparison of the Elliott Model and the Data

The following approaches are used in determining the model parameters.

1. Using the transition probabilities obtained from the Gilbert model for the error gap distribution, find the value of k that provides a best fit to the $P(m,n)$ data.

2. Find parameter values of P_{12}, P_{21}, h, and k that best fit the block error data.

The error gap distributions used in determining P_{12}, P_{21}, and h are the same as in section 3. For convenience, the values of J, L, and A are reproduced in Table 4.1. Table 4.2 shows the corresponding derived parameters used to determine $P(m,n)$. The plots resulting from these parameters are shown in Figures 4.3 to 4.11. With

Table 4.2: Elliott Model Parameters for the Block Error Probabilities Based on Best Fit to the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	h	P_{12}	P_{21}	k
070901	II	0.678	1.75×10^{-5}	9.62×10^{-4}	0.999999
070903	III	0.589	6.97×10^{-5}	2.05×10^{-4}	0.999996
070905	II	0.622	9.31×10^{-5}	1.70×10^{-4}	0.999984
070906	III	0.596	3.47×10^{-5}	2.77×10^{-4}	0.999998
070907	II	0.622	5.26×10^{-5}	4.71×10^{-4}	0.999997
070912	III	0.562	9.87×10^{-5}	1.61×10^{-4}	0.999994
070914	III	0.611	2.65×10^{-4}	2.02×10^{-4}	0.999999
071016	II	0.675	8.10×10^{-6}	2.18×10^{-4}	0.999999
071017	II	0.750	6.52×10^{-6}	1.35×10^{-4}	0.999997
072405	II	0.985	8.62×10^{-5}	2.24×10^{-5}	0.517587
072406	II	0.629	4.23×10^{-5}	6.35×10^{-4}	0.999999
072407	II	0.567	7.44×10^{-5}	2.02×10^{-4}	0.999994
072408	II	0.629	2.41×10^{-5}	4.22×10^{-4}	0.999994
072409	II	0.629	2.87×10^{-5}	1.16×10^{-3}	0.999999
072410	II	0.603	1.43×10^{-4}	2.00×10^{-4}	0.999999
072411	II	0.622	4.73×10^{-5}	7.74×10^{-4}	0.999999
072412	II	0.598	7.00×10^{-5}	2.88×10^{-4}	0.999996

the addition of the parameter k alters the values and bring the model's block error probabilities slightly closer to that of the data, it does not have enough of an effect to provide a good fit. While there is slight improvement over the Gilbert model, it is not enough to make a significant difference in the curves. Again the values of $\overline{P}(m,n)$ are approximated well only for the smaller values of m.

As in chapter 3, there are model parameters that do fit the block error statistics better than those values derived from the parameters used to model the error gap distribution curves. These values for the Elliott model are given in Table 4.3 with the corresponding derived parameters for the error gap distribution given in Table 4.4.

Figures 4.12, 4.14, 4.16, 4.18, and 4.20 show the results of trying to find a best fit to the curves for $P(m,n)$. The corresponding error gap distribution curves are given in Figures 4.13, 4.15, 4.17, 4.19, and 4.21.

These plots show that choosing appropriate parameters for the Elliott model results in slightly better performance than the Gilbert model. The resulting

Table 4.3: Elliott Model Parameters for Block Error Probabilities Based on Best Fit to the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	P_{12}	P_{21}	h	k
070901	II	3.16×10^{-5}	1.84×10^{-3}	0.515	0.995625
070903	III	2.87×10^{-4}	7.35×10^{-4}	0.502	0.999998
070905	II	4.22×10^{-4}	5.51×10^{-4}	0.497	0.999999
070906	III	8.35×10^{-5}	4.25×10^{-4}	0.508	0.999999
070907	II	1.38×10^{-4}	1.18×10^{-3}	0.512	0.999999
070912	III	2.31×10^{-4}	3.74×10^{-4}	0.494	0.999999
070914	III	3.84×10^{-3}	4.01×10^{-4}	0.498	0.999999
071016	II	3.25×10^{-3}	5.40×10^{-5}	0.520	0.999999
071017	II	1.15×10^{-4}	4.40×10^{-4}	0.520	0.999999
072405	II	2.94×10^{-4}	1.33×10^{-3}	0.518	0.999999
072406	II	1.19×10^{-4}	2.35×10^{-3}	0.523	0.999999
072407	II	1.53×10^{-4}	3.96×10^{-4}	0.491	0.999998
072408	II	2.90×10^{-5}	2.72×10^{-4}	0.521	0.999998
072409	II	6.00×10^{-6}	1.31×10^{-4}	0.496	0.999977
072410	II	4.48×10^{-4}	4.85×10^{-4}	0.497	0.999999
072411	II	2.16×10^{-3}	2.70×10^{-5}	0.517	0.999999
072412	II	1.39×10^{-4}	6.01×10^{-4}	0.508	0.999999

curves are very close to the values of $\overline{P}(m,n)$. Also, as seen in the Figures, parameters that give the best fit to $\overline{P}(m,n)$ result in fits to the error gap distribution curves equivalent to the Gilbert model, as is expected.

The effect of adding the parameter k to the Gilbert model in forming the Elliott model only provides a very small improvement in modeling $\overline{P}(m,n)$ given the error gap distribution as described previously. This implies that the model is limited in how well it can characterize this type of data for both the error gap distribution and the block error probabilities using the methods outlined by Gilbert and Elliott. However, if the derivation of the transition probability and error probability values from the error gap distribution are not used and the model parameter values are chosen based on minimizing the error between $\overline{P}(m,n)$ and the derived block error probabilities, it is possible to find parameters that model the block error statistics and still adequately characterize the error gap distribution. This indicates that for both Gilbert and Elliott models, the method of parameter estimation based on error gap distribution information used to derive the transition

Table 4.4: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	J	L	A
070901	II	0.999968	0.514388	0.003790
070903	III	0.999713	0.501330	0.001480
070905	II	0.999578	0.496871	0.001100
070906	III	0.999916	0.508292	0.000865
070907	II	0.999862	0.512238	0.002430
070912	III	0.999769	0.494674	0.000741
070914	III	0.996158	0.497946	0.000808
071016	II	0.996747	0.520330	0.000114
071017	II	0.999885	0.520071	0.000917
072405	II	0.999706	0.517864	0.002760
072406	II	0.999881	0.522578	0.004910
072407	II	0.999847	0.491140	0.000779
072408	II	0.999971	0.520914	0.000568
072409	II	0.999994	0.496828	0.000260
072410	II	0.999552	0.497099	0.000966
072411	II	0.997836	0.517698	0.000056
072412	II	0.999861	0.508595	0.001220

probabilities has its limitations in modeling the data on the channel. But if it is desired to model only $\overline{P}(m,n)$ or the error gap distribution alone, the models can be used successfully.

In summary, for just one of the two curves, the error gap distribution or $\overline{P}(m,n)$, this model does a good job characterizing the error gap distribution, particularly the longer gap distributions. It also does an excellent job of characterizing the block error probabilities.

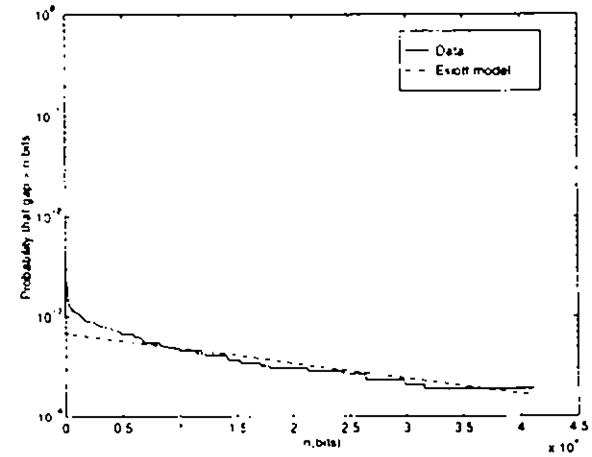

Figure 4.2: Error Gap Distribution Using the Elliott model with Parameters Chosen to Match the Error Gap Distribution for Run 070906 for $\eta = -6dB$.

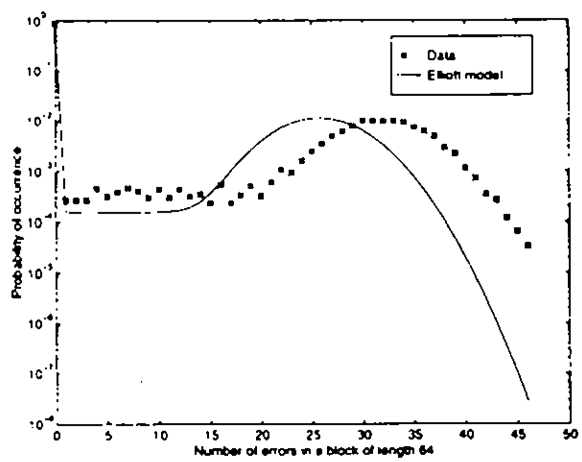

Figure 4.3: Block error probabilities using the Elliott model with parameters chosen to match the error gap distribution for Run 070906 for $\eta = -6dB$.

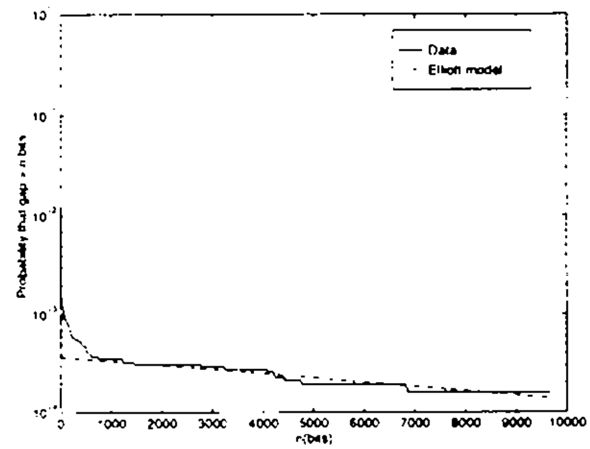

Figure 4.4: Error Gap Distribution Using the Elliott model with Parameters Chosen to Match the Error Gap Distribution for Run 070912 for $\eta = -6dB$.

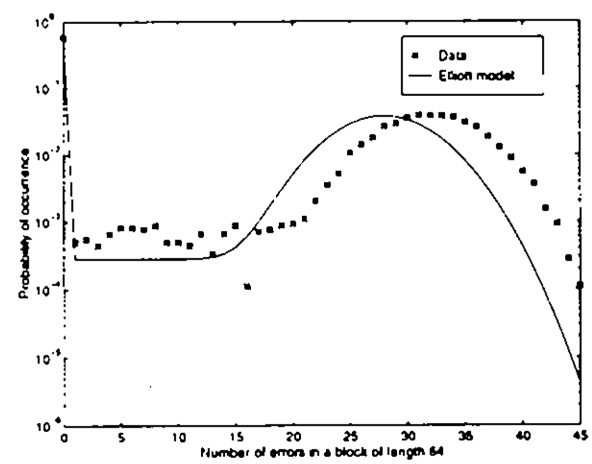

Figure 4.5: Block error probabilities using the Elliott model with parameters chosen to match the error gap distribution for Run 070912 for $\eta = -6dB$.

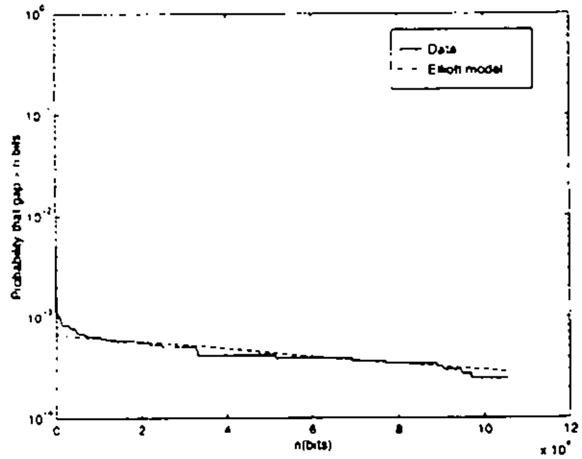

Figure 4.6: Error Gap Distribution Using the Elliott model with Parameters Chosen to Match the Error Gap Distribution for Run 071016 for $\eta = -6dB$.

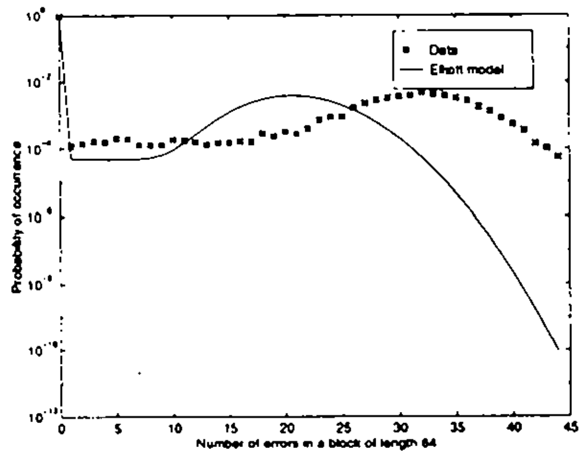

Figure 4.7: Block error probabilities using the Elliott model with parameters chosen to match the error gap distribution for Run 071016 for $\eta = -6dB$.

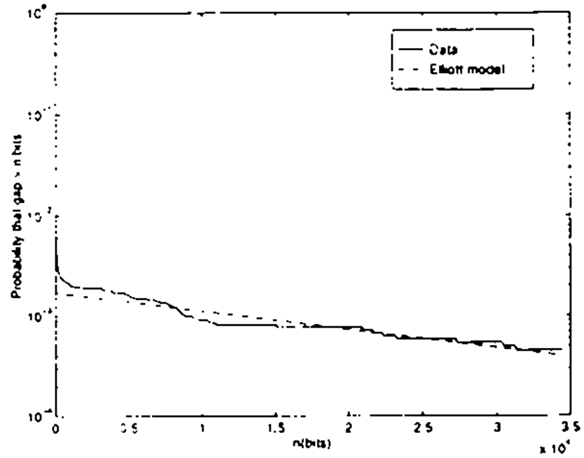

Figure 4.8: Error Gap Distribution Using the Elliott model with Parameters Chosen to Match the Error Gap Distribution for Run 072406 for $\eta = -6dB$.

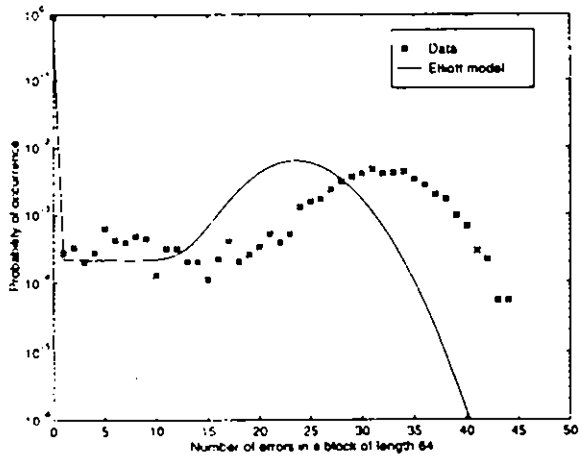

Figure 4.9: Block error probabilities using the Elliott model with parameters chosen to match the error gap distribution for Run 072406 for $\eta = -6dB$.

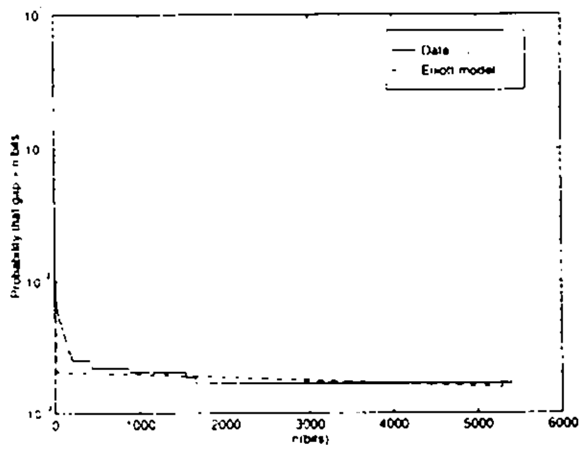

Figure 4.10: Error Gap Distribution Using the Elliott model with Parameters Chosen to Match the Error Gap Distribution for Run 072411 for $\eta = -6dB$.

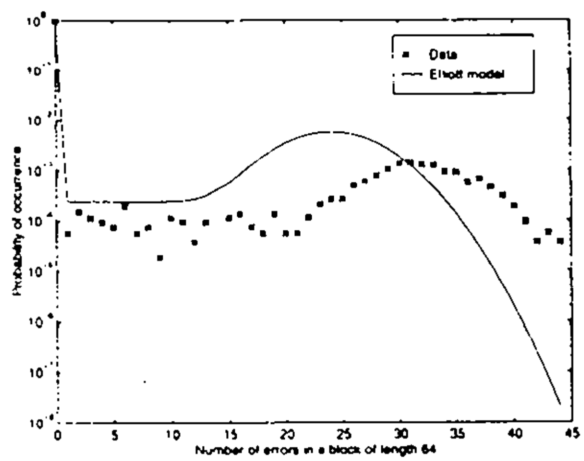

Figure 4.11: Block error probabilities using the Elliott model with parameters chosen to match the error gap distribution for Run 072411 for $\eta = -6dB$.

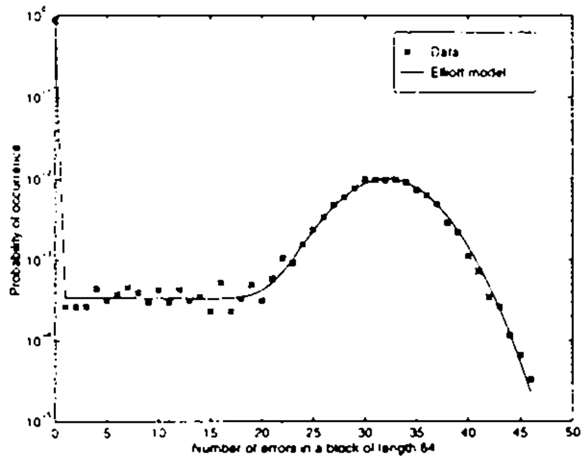

Figure 4.12: Block Error Probabilities Using the Elliott Model with Parameters Chosen to Match the Block Error Probabilities for Run 070906 for $\eta = -6dB$.

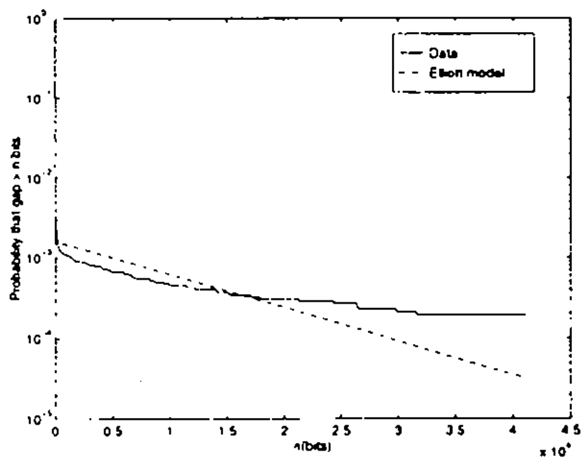

Figure 4.13: Error gap distribution using the Elliott model with parameters chosen to match the block error probabilities for Run 070906 for $\eta = -6dB$.

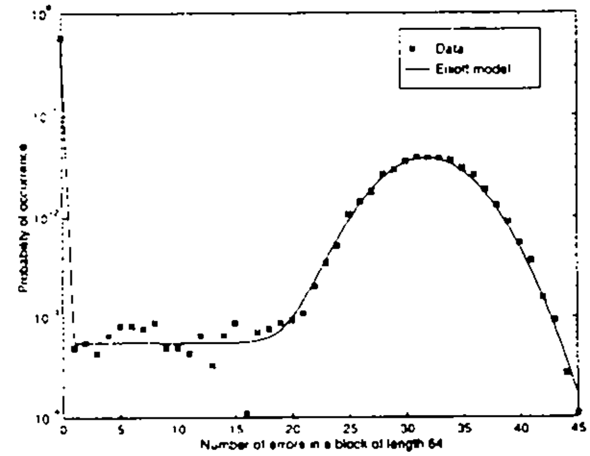

Figure 4.14: Block Error Probabilities Using the Elliott Model with Parameters Chosen to Match the Block Error Probabilities for Run 070912 for $\eta = -6dB$.

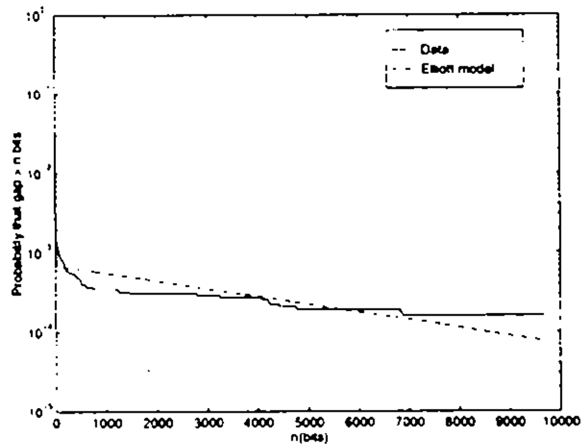

Figure 4.15: Error gap distribution using the Elliott model with parameters chosen to match the block error probabilities for Run 070912 for $\eta = -6dB$.

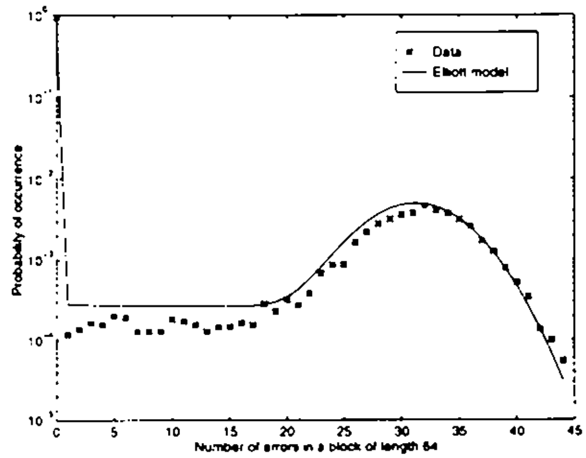

Figure 4.16: Block Error Probabilities Using the Elliott Model with Parameters Chosen to Match the Block Error Probabilities for Run 071016 for $\eta = -6dB$.

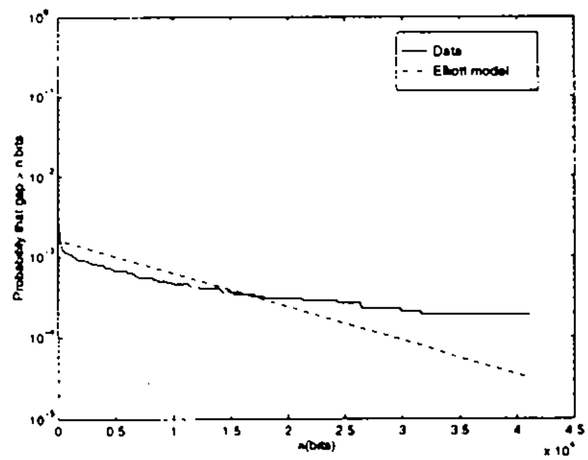

Figure 4.17: Error gap distribution using the Elliott model with parameters chosen to match the block error probabilities for Run 071016 for $\eta = -6dB$.

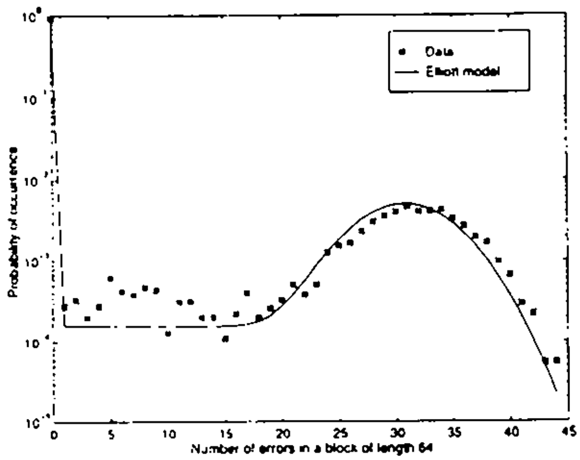

Figure 4.18: Block Error Probabilities Using the Elliott Model with Parameters Chosen to Match the Block Error Probabilities for Run 072406 for $\eta = -6dB$.

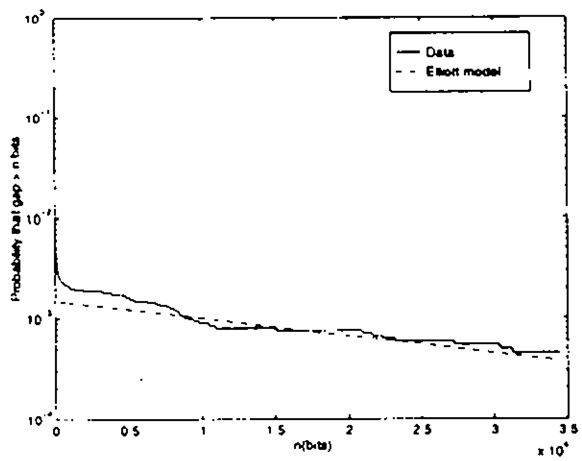

Figure 4.19: Error gap distribution using the Elliott model with parameters chosen to match the block error probabilities for Run 072406 for $\eta = -6dB$.

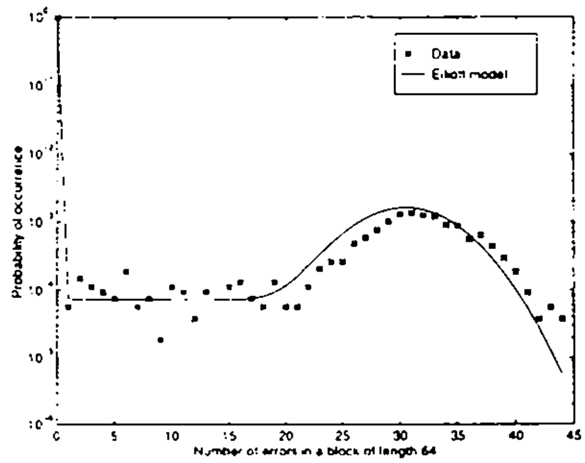

Figure 4.20: Block Error Probabilities Using the Elliott Model with Parameters Chosen to Match the Block Error Probabilities for Run 072411 for $\eta = -6dB$.

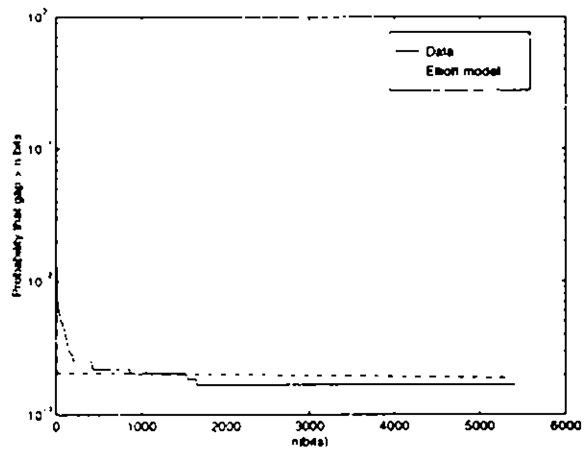

Figure 4.21: Error gap distribution using the Elliott model with parameters chosen to match the block error probabilities for Run 072411 for $\eta = -6dB$.

Chapter 5

THE MCCULLOUGH MODEL

The McCullough model is the generalization of both the Gilbert and Elliott models. In these earlier models, the transition probability between states is independent of whether an error occurs or not. For this model, this assumption is removed so that the transition between states is allowed only after an error occurs. With this added structure, the resulting model is termed the "binary regenerative channel" [7].

5.1 The Model

McCullough's general Markov channel model is shown in Figure 5.1 where

$$s(n) = (1 \text{ or } 2)$$

$$S_{s(n)} = \text{error state for the } n\text{th error digit}$$

$$Z = \text{the error digit, 1 for error and 0 for no error}$$

$$p_{ij} = \text{Prob}\{S_{s(n)} = j | S_{s(n-1)} = i, Z_{n-1} = 0\}$$

$$q_{ij} = \text{Prob}\{S_{s(n)} = j | S_{s(n-1)} = i, Z_{n-1} = 1\}$$

$$P_i = \text{Prob}\{Z_n = 1 | S_{s(n)} = i\} = \text{average error probability of state } i.$$

This model is a generalization of the previous models. In this model, the transition probabilities between states now depend on whether or not an error occurs in that state. By setting $p_{12} = q_{12}$ and $p_{21} = q_{21}$, the model reduces to the Elliott model, where state transitions are independent of whether or not an error occurs.

By setting $p_{ij} = \delta_{ij}$, the Kronecker delta, the "binary regenerative channel" is formed, which allows transitions between states only after an error bit has occurred. This is based on the idea that the only information available about the channel comes from the occurrence of errors [7]. Therefore, switching between states is said to occur only after an error. If this assumption is made, then the model reduces to the one in Figure 5.2.

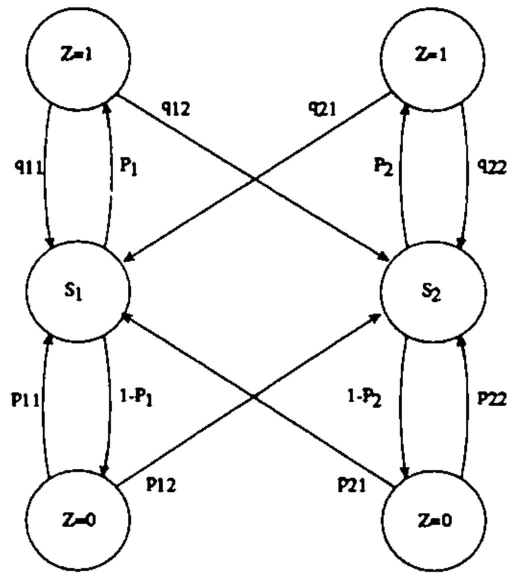

Figure 5.1: General two-state Markov Model.

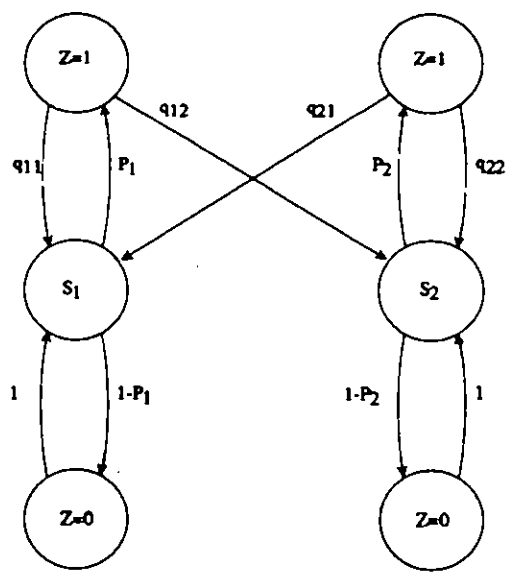

Figure 5.2: McCullough's Binary Regenerative Channel.

5.2 The Error Gap Distribution and Block Error Probabilities

The error gap distribution for the McCullough model, $u(n)$, is the sum of weighted exponentials, similar in form to the earlier models. Following the method in [7], this is derived below.

First, the basic equations for an independent error process are derived. Let P be the probability that any digit is in error. The error sequence then contains a one with probability P and a zero with probability $1 - P$. Given an error, the probability that the next error occurs on the kth bit is

$$\begin{aligned} p(k) &= \mathrm{Prob}\{0^{k-1}1|1\} \\ &= \mathrm{Prob}\{0^{k-1}1\} \\ &= P(1-P)^{k-1}. \end{aligned}$$

The average error separation is

$$\begin{aligned} \overline{k} &= \sum_{k=1}^{\infty} kp(k) \\ &= P\sum_{k=1}^{\infty} k(1-P)^{k-1} \\ &= \frac{1}{P}. \end{aligned}$$

The probability that the number of error free bits is greater than n is given by the cumulative distribution

$$\begin{aligned} u(n) &= \mathrm{Prob}\{k > n\} \\ &= 1 - \sum_{k=1}^{n} p(k) \\ &= 1 - P\sum_{k=1}^{n}(1-P)^{k-1} \\ &= (1-P)^{n}. \end{aligned}$$

Now, let Q_j, $j = 1, 2$, be the unconditional probability of being in state j at the first digit following an error. Using the above results for independent error processes yields

$$
\begin{aligned}
p(k) &= Q_1P_1(1-P_1)^{k-1} + Q_2P_2(1-P_2)^{k-1} && (5.1)\\
\bar{k} &= \frac{Q_1}{P_1} + \frac{Q_2}{P_2} \equiv \frac{1}{P_e} && (5.2)\\
u(n) &= Q_1(1-P_1)^n + Q_2(1-P_2)^n && (5.3)
\end{aligned}
$$

for the two state model where P_e is the overall average error rate. As seen in earlier models, $u(n)$ in equation (5.3) is the error gap distribution.

The block error probabilities $P(m,n)$ are found using

$$P(m,n) = A_1(m,n) + A_2(m,n) \tag{5.4}$$

where $A_i(m,n)$ is the probability that m errors have occurred in n bits and that the process is in state i after n bits. Using the following set of recurrence relations, A_1 and A_2 are determined by

$$
\begin{aligned}
A_1(m,n) = A_1(m,n-1)(1-P_1) + A_1(m-1,n-1)P_1q_{11}\\
+ A_2(m-1,n-1)P_2q_{21}
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
A_2(m,n) = A_2(m,n-1)(1-P_2) + A_2(m-1,n-1)P_2q_{22}\\
+ A_1(m-1,n-1)P_1q_{12}.
\end{aligned} \tag{5.6}
$$

The initial conditions for A_1 and A_2 are

$$
\begin{aligned}
A_1(0,0) &= Q_1\\
A_2(0,0) &= Q_2\\
A_i(m,n) &= 0 \text{ for } m > n.
\end{aligned}
$$

As noted in (5.5) and (5.6), the probabilities q_{ij} must be known in order to determine $P(m,n)$. The transition probabilities between the states are determined from the data. To determine these, each state is first classified as a burst-error state or a random-error state. To determine whether a particular error is produced by a burst or a random-error state, a threshold value k is chosen. If the gap between errors is less than k, then the error is considered to be produced by the burst state. If the gap between errors is greater than k, then the error is considered to be produced by the random error state.

In McCullough's model, a value of $k = 10$ represented the threshold between burst error and random error states. This value will be used to check the ACTS data in the sample runs.

5.3 Comparison of the McCullough Model and the Data

To determine the applicability of this model, the following method is used:

1. Choose a threshold k which determines whether the error state causing the error is due to a "burst error state" or a "random error state."

2. Get the error separation for successive error gaps in the data sequence, and classify it as either a burst error or a random error.

3. For each state i, get the average error separation, $\overline{k_i} = 1/P_i$.

4. From the sequence of states obtain Q_i and the transition probabilities q_{ij}.

5. Use the above parameters to obtain the error gap distribution and the block error probabilities.

Based on the threshold value $k = 10$, which McCullough uses to characterize his data, the model parameters are derived and placed in Table 5.1. Using these values to plot $u(n)$ in comparison to the measured data, the results of Figures 5.3 through 5.11 show that this model using $k = 10$ does not come very close to matching the curves. The two curves diverge from each other, and the model based on the given parameters yields much lower probabilities than the data produces. Using these parameters to determine $P(m,n)$, the plots show a degradation in performance for this value of k when compared to the Gilbert and Elliott models. In particular, the overall probability values obtained using the model are much higher than the data values. The results are seen in Figures 5.4, 5.6, 5.8, 5.10, and 5.12. It is encouraging to see that the values of $P(m,n)$ produced by using this model take on the general form of $\overline{P}(m,n)$ for the different runs, in particular, the characteristic "hump"

Table 5.1: McCullough Model Parameters Based On $k = 10, \eta = -6dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.991786	0.013699	0.991740	0.008260	0.510873	0.000065
070903	III	0.997688	0.005435	0.997680	0.002320	0.504467	0.000592
070905	II	0.997231	0.008115	0.997216	0.002780	0.502659	0.000930
070906	III	0.996593	0.002817	0.996600	0.003400	0.502499	0.000201
070907	II	0.995652	0.004250	0.995652	0.000435	0.505040	0.000258
070912	III	0.998002	0.004032	0.997998	0.002002	0.503224	0.000779
070914	III	0.996862	0.005556	0.996854	0.003150	0.503145	0.001743
071016	II	0.995449	0.005464	0.995445	0.004555	0.501682	0.000109
071017	II	0.994649	0.011628	0.994615	0.005390	0.501043	0.000119
072405	II	0.992029	0.003451	0.992065	0.007940	0.504465	0.000631
072406	II	0.993525	0	0.993567	0.006433	0.502845	0.000169
072407	II	0.997904	0.001727	0.997905	0.002100	0.503620	0.000412
072408	II	0.995127	0.013575	0.995084	0.004920	0.500528	0.000138
072409	II	0.992682	0.012195	0.992646	0.007350	0.501537	0.000048
072410	II	0.997265	0.005208	0.997258	0.002740	0.502578	0.001132
072411	II	0.994264	0.014286	0.994214	0.005786	0.502906	0.000047
072412	II	0.996851	0.005367	0.996844	0.003160	0.503137	0.000349

seen in all of the curves, even though the state transitions have been restricted to occur only after errors.

The most probable cause of the poor match of the model to the data is in the choice of k. While McCullough uses the value of $k = 10$ in his model, there is likely a better choice when analyzing other types of data, such as the ACTS data. However, there is no set method for choosing an appropriate value of k given by McCullough. This is one of the difficulties in using this model, and will be addressed shortly.

The parameters of P and Q that match the error gap distribution the closest are given by Table 5.2. Also included in the Table are the transition probabilities q_{11} and q_{22}. Because there are an infinite number of possible transition probabilities to choose that yield the same steady state probabilities $Q1$ and $Q2$, q_{11} and q_{22} are chosen to give the best fit to the $\overline{P}(m,n)$ curve.

Table 5.2: McCullough Model Parameters Chosen to Match Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.999876	0.958639	0.997023	0.002977	0.323145	0.000017
070903	III	0.999501	0.000283	0.999502	0.000498	0.411036	0.000069
070905	II	0.999549	0.000018	0.999549	0.000045	0.377832	0.000093
070906	III	0.999704	0.568000	0.999315	0.000684	0.404366	.000034
070907	II	0.999682	0.777641	0.998756	0.001244	0.378418	0.000052
070912	III	0.999632	0.000035	0.999632	0.000368	0.438203	.000098
070914	III	0.999478	0.000054	0.999478	0.000521	0.388769	0.000265
071016	II	0.999877	0.817678	0.999331	0.000669	0.325221	.000008
071017	II	0.999891	0.799415	0.999461	0.000539	0.250000	0.000006
072405	II	0.997341	0.828900	0.999948	0.000051	0.001900	0.000005
072406	II	0.999722	0.837577	0.998291	0.001709	0.371583	.000042
072407	II	0.999603	0.150029	0.999533	0.000466	0.432813	0.000074
072408	II	0.999812	0.834972	0.998865	0.001135	0.371583	0.000024
072409	II	0.999908	0.970712	0.996886	0.003114	0.371290	0.000028
072410	II	0.999494	0.000025	0.999494	0.000505	0.396973	0.000143
072411	II	0.999931	0.966222	0.997956	0.002044	0.378614	.000047
072412	II	0.999579	0.412376	0.999284	0.000715	0.402051	0.000069

The resulting plots are seen in Figures 5.13, 5.15, 5.17, 5.19 and 5.21. Using the corresponding transition probabilities in Table 5.2, the block error probabilities are plotted in Figures 5.14, 5.16, 5.18, 5.20, and 5.22. These plots show that matching the error gap distribution curves does not result in the matching of the block error probabilities. In fact $P(m,n)$ values obtained in this manner now do worse than those obtained using both Gilbert and Elliott's models.

The parameters that do fit the $\overline{P}(m,n)$ curves the best were found and are presented in Table 5.3.

The five sample run block error probability curves are shown in Figures 5.23, 5.25, 5.27, 5.29, and 5.31. with the corresponding error gap distribution curves below in Figures

As would be expected, the McCullough model does an equivalent job in modeling the block error statistics when compared to the Gilbert or Elliott Models. With it being more restrictive in nature (with respect to the transitions being allowed only after an error), some of the curves show a little worse performance.

Table 5.3: McCullough Model Parameters Chosen to Match Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.990553	0.233954	0.016470	0.983530	0.503795	3.22×10^{-10}
070903	III	.998084	0.175897	0.348031	0.651969	0.501443	2.72×10^{-10}
070905	II	0.997680	0.034927	0.425552	0.574448	0.506998	2.89×10^{-10}
070906	III	0.996833	0.225946	0.110316	0.889684	0.502786	2.56×10^{-10}
070907	II	0.995466	0.129568	0.089953	0.910047	0.509191	2.46×10^{-12}
070912	III	0.9986169	0.159234	0.402314	0.597686	0.498854	2.54×10^{-10}
070914	III	0.998013	0.109217	0.545000	0.455000	0.498464	2.54×10^{-10}
071016	II	0.9965864	0.1072376	0.048623	0.951376	0.498145	3.00×10^{-10}
071017	II	0.995744	0.327693	0.045394	0.954606	0.497824	3.64×10^{-10}
072405	II	0.992538	0.231381	0.151061	0.848939	0.503814	2.29×10^{-10}
072406	II	0.999636	6.93×10^{-11}	0.769358	0.230642	0.497160	0.000155
072407	II	0.998926	0.165302	0.401982	0.598018	0.503452	2.54×10^{-10}
072408	II	0.998256	0.112784	0.121366	0.878634	0.489798	3.30×10^{-16}
072409	II	0.991515	0.054639	0.014099	0.985901	0.503670	3.71×10^{-10}
072410	II	0.998061	0.158672	0.485162	0.514838	0.504510	1.97×10^{-10}
072411	II	0.994407	0.296454	0.014720	0.98528	0.507189	3.00×10^{-10}
072412	II	0.998737	0.306878	0.374730	0.625270	0.488872	4.61×10^{-12}

The restrictions on the transition probabilities of the binary regenerative channel model , however, result in a worse match between the model and the data for the corresponding error gap distribution curves.

The effects on the error gap distribution, using the parameters optimizing $P(m,n)$, are shown in Figures 5.24, 5.26, 5.28, 5.30, and 5.32. From these plots it appears that the McCullough model for the error gap distribution in this case is grossly inaccurate for all values of n. In some of these plots, the curve representing the model is close to the top of the plot and is almost completely flat. This is due to the low probability of error in one of the states for the best fit for the model. This is a more extreme example of how modeling $\overline{P}(m,n)$ well reduces modeling accuracy for the error gap distribution. As seen in the earlier models, either $\overline{P}(m,n)$ or the error gap distribution could be modeled with fair accuracy, and fitting one curve resulted in a poorer fit of the other curve. In this model, $\overline{P}(m,n)$ is modeled with high accuracy at the expense of modeling the error gap distribution. In this respect, the McCullough model is inferior to the Gilbert and Elliott models.

As mentioned earlier, the model parameters obtained are highly dependent on the threshold value of k chosen. Due to this influence of the model parameters on the value of k, an attempt was made to find the best value of k for each run. Using the parameters that generated the best fit to the block error probabilities in Table 5.3, the values of k were evaluated by testing possible values of k from 10 up to 10000 and comparing the $P(m,n)$ values generated for each value of k to the $\overline{P}(m,n)$ values. The best k was then chosen on the basis of mean square error between the two curves. The results of this search are found in Table 5.4. The corresponding plots are found in Figures 5.33, 5.35, 5.37, 5.39, and 5.42. These plots show that even if an appropriate threshold is chosen for a particular run, the resulting model curves still may not be able to produce a close fit to $\overline{P}(m,n)$. The same is true for the corresponding error gap distribution plots as seen in these Figures. This is due to a mismatch in the model and the data. The model assumes that there is a bursty error producing state and a random error producing state. However in the synthetic error sequence, there are no random errors produced when the received pilot power levels are greater than the fading threshold η. The McCullough model requires a non-zero probability of error in the random error producing state. Otherwise, there is no way to leave that state once it is entered.

While the McCullough can provide good fits to the block error probabilities, it does not characterize corresponding fits to the error gap distributions. The added structure of this model is too restrictive to allow accurate characterization of the channel. The choice of a threshold for a particular run also makes the model more complex to use since that threshold must be known beforehand, and as is seen in the curves mentioned earlier, still does not guarantee a good fit. This is the biggest disadvantage of using this model. In the next and final model, the Fritchman model, a different type of Markov chain model is presented in which the states in the Markov chain represent individual bits instead of binary symmetric channels.

Table 5.4: McCullough Model Parameters Using Best $k, \eta = -6dB$.

RUN	CAT.	k	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	740	0.998736	0.076923	0.998623	0.001380	0.355509	0.000011
070903	III	10	0.997688	0.005435	0.997680	0.002320	0.504467	0.000592
070905	II	20	0.998709	0.013436	0.998693	0.001307	0.498883	0.000439
070906	III	430	0.999414	0.015873	0.999405	0.000595	0.465702	0.000036
070907	II	460	0.999257	0.012658	0.999248	0.000752	0.447668	0.000045
070912	III	20	0.999229	0.010309	0.999222	0.000778	0.499994	0.000305
070914	III	20	0.998785	0.000000	0.998786	0.001214	0.498143	0.000682
071016	II	1450	0.999613	0.030303	0.999601	0.000399	0.434376	0.000010
071017	II	730	0.999529	0.021277	0.999519	0.000481	0.431400	0.000011
072405	II	210	0.998469	0.007092	0.998460	0.001540	0.451716	0.000125
072406	II	740	0.998938	0.0000000	0.998939	0.001061	0.437516	0.000028
072407	II	40	0.999336	0.000000	0.999337	0.000663	0.499340	0.000131
072408	II	2000	0.999284	0.000000	0.999285	0.000715	0.418303	0.000020
072409	II	560	0.998636	0.058824	0.998547	0.001453	0.399390	0.000009
072410	II	20	0.998863	0.008299	0.998854	0.001146	0.498438	0.000477
072411	II	1660	0.999615	0.000000	0.999615	0.000385	0.452093	0.000044
072412	II	300	0.999440	0.000000	0.999440	0.000560	0.478072	0.000063

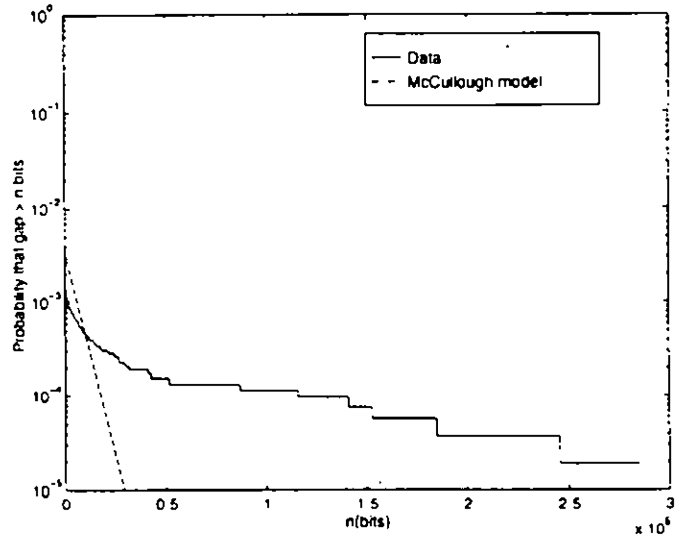

Figure 5.3: Error Gap Distribution for Threshold of $k = 10$ for Run 070906 for $\eta = -6dB$.

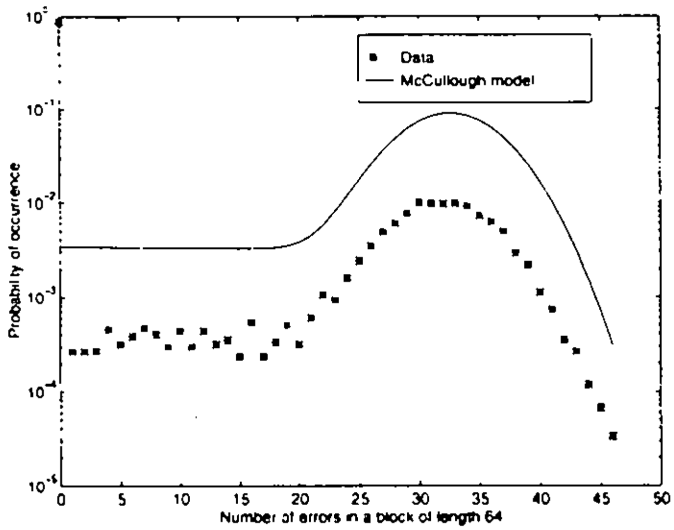

Figure 5.4: Block Error Probabilities for Threshold of $k = 10$ for Run 070906 for $\eta = -6dB$.

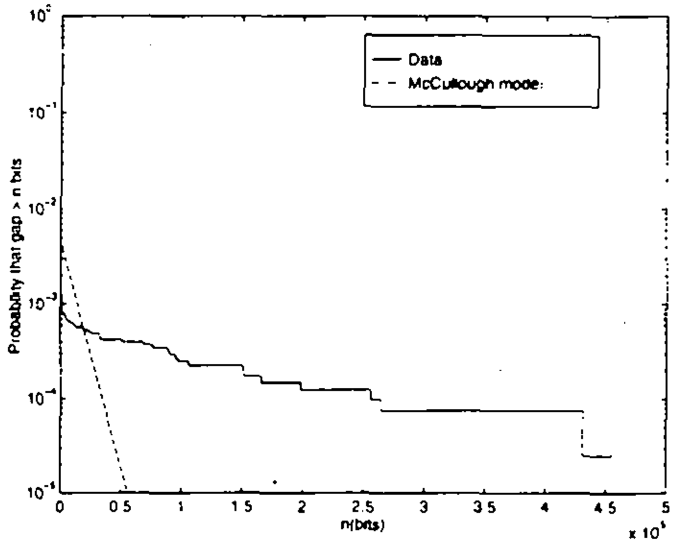

Figure 5.5: Error Gap Distribution for Threshold of $k = 10$ for Run 070912 for $\eta = -6dB$.

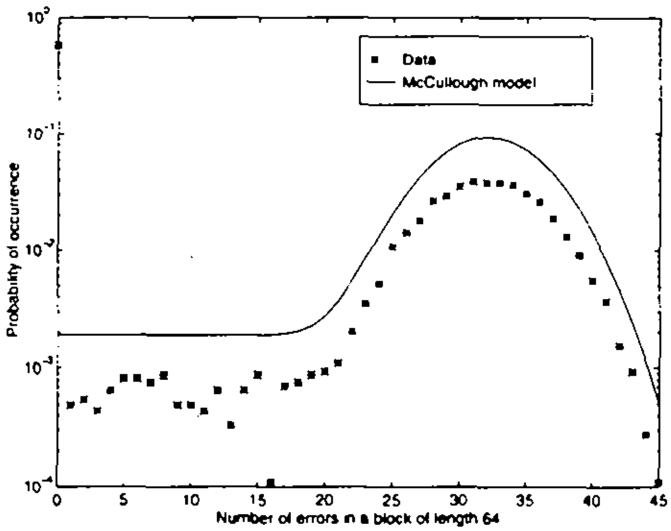

Figure 5.6: Block Error Probabilities for Threshold of $k = 10$ for Run 070912 for $\eta = -6dB$.

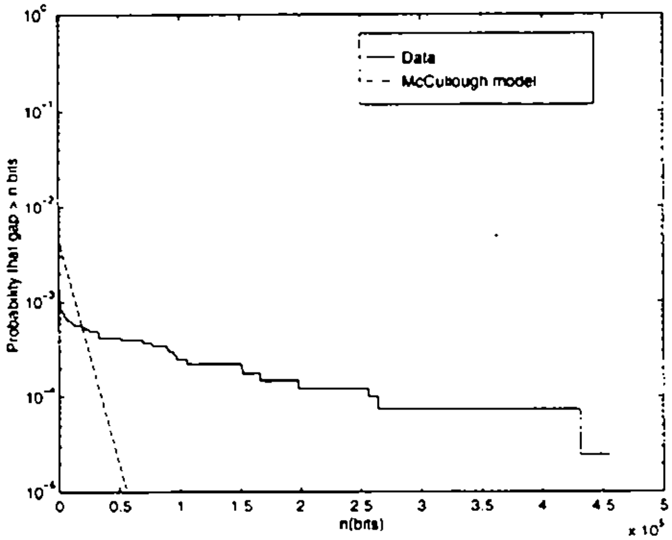

Figure 5.7: Error Gap Distribution for Threshold of $k = 10$ for Run 071016 for $\eta = -6dB$.

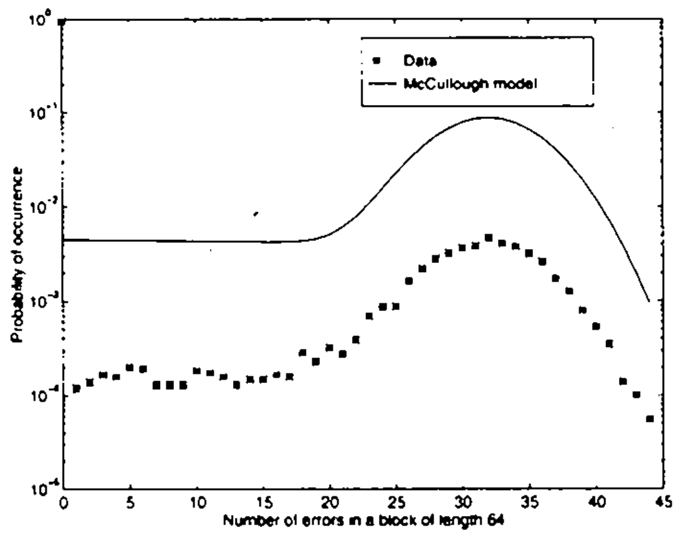

Figure 5.8: Block Error Probabilities for Threshold of $k = 10$ for Run 071016 for $\eta = -6dB$.

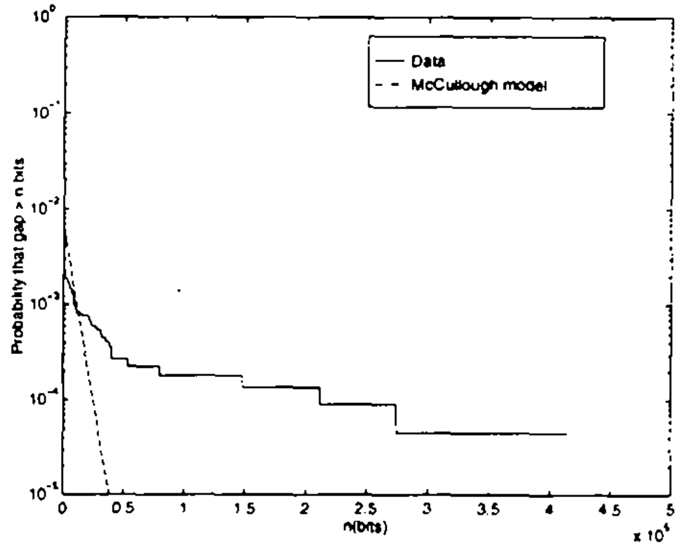

Figure 5.9: Error Gap Distribution for Threshold of $k = 10$ for Run 072406 for $\eta = -6dB$.

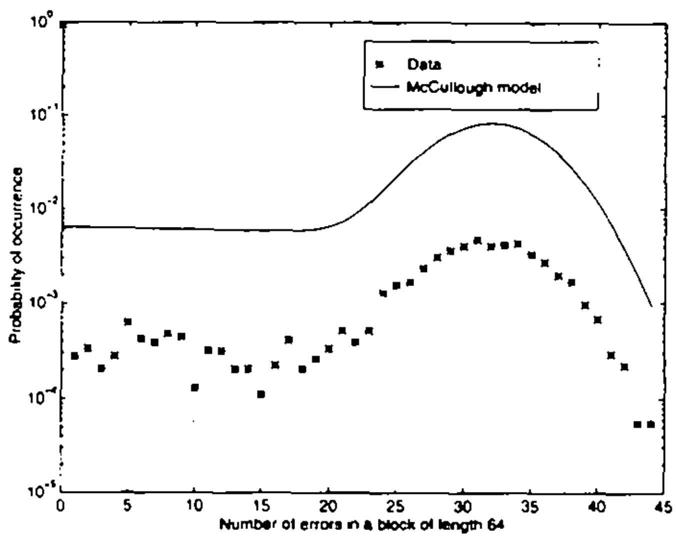

Figure 5.10: Block Error Probabilities for Threshold of $k = 10$ for Run 072406 for $\eta = -6dB$.

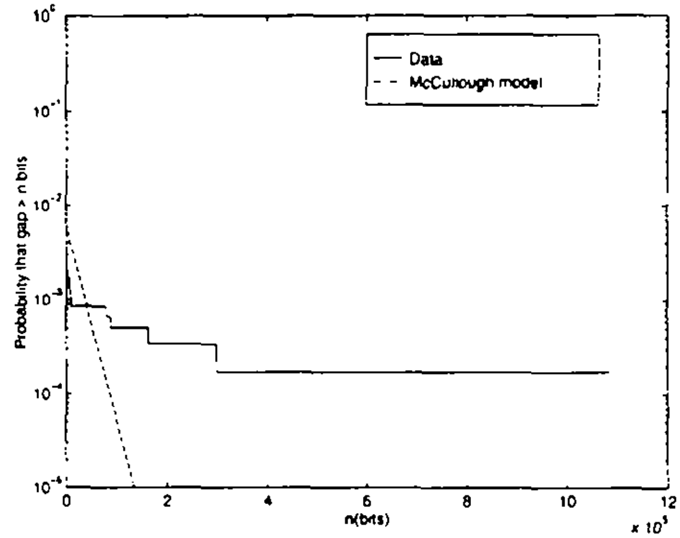

Figure 5.11: Error Gap Distribution for Threshold of $k = 10$ for Run 072411 for $\eta = -6dB$.

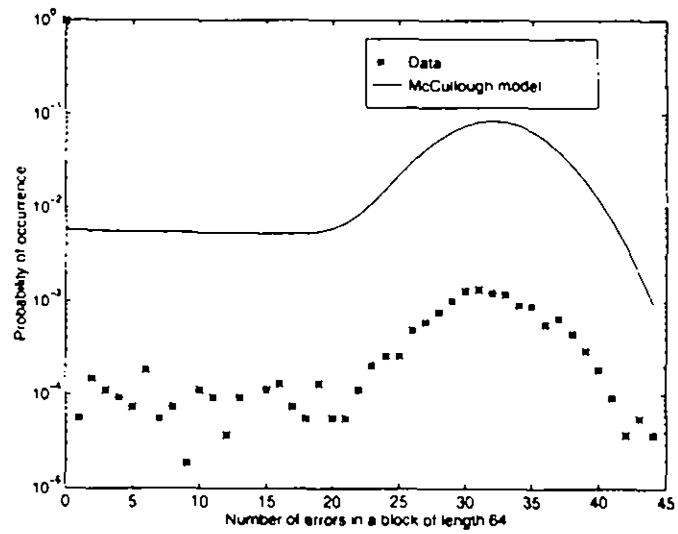

Figure 5.12: Block Error Probabilities for Threshold of $k = 10$ for Run 072411 for $\eta = -6dB$.

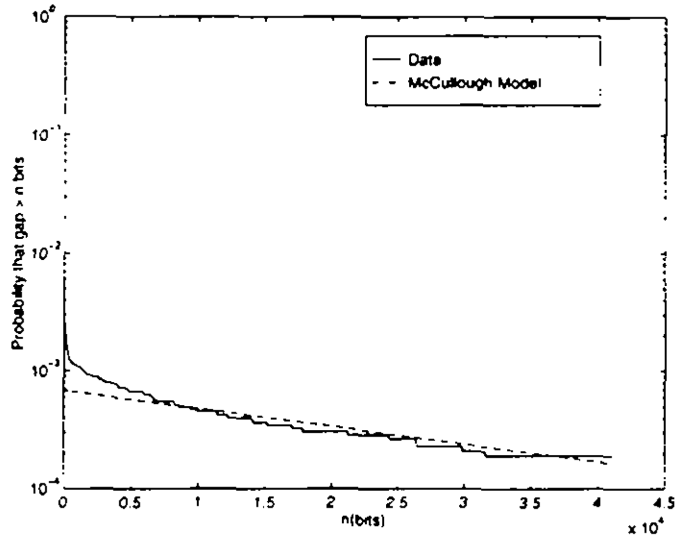

Figure 5.13: Error Gap Distribution Using Parameters Chosen to Match the Error Gap Distribution for Run 070906 for $\eta = -6dB$.

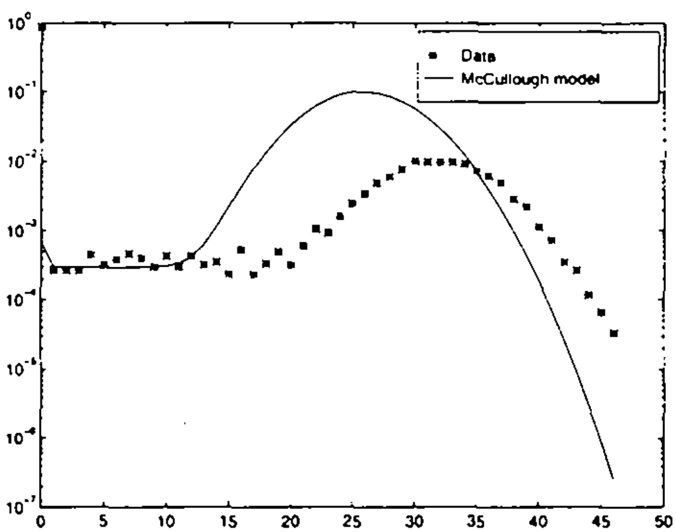

Figure 5.14: Block Error Probabilities Using Parameters Chosen to Match the Error Gap Distribution for Run 070906 for $\eta = -6dB$.

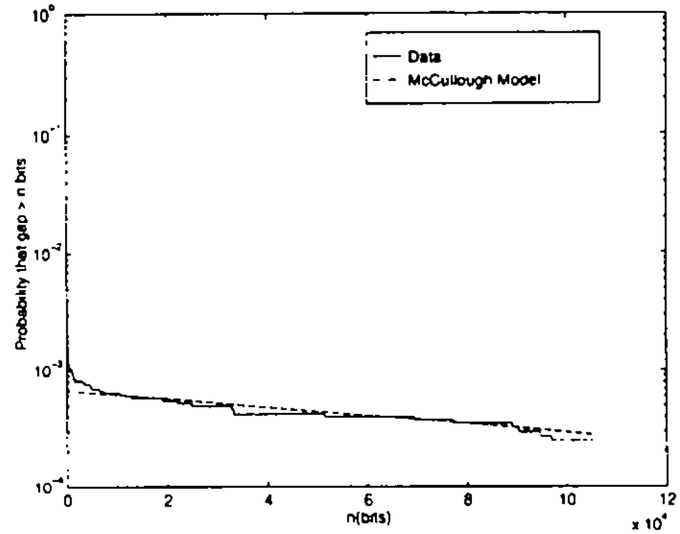

Figure 5.15: Error Gap Distribution Using Parameters Chosen to Match the Error Gap Distribution for Run 070912 for $\eta = -6dB$.

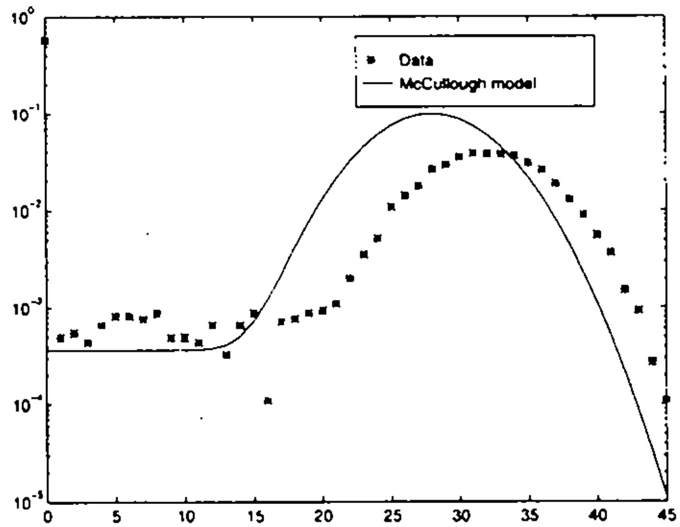

Figure 5.16: Block Error Probabilities Using Parameters Chosen to Match the Error Gap Distribution for Run 070912 for $\eta = -6dB$.

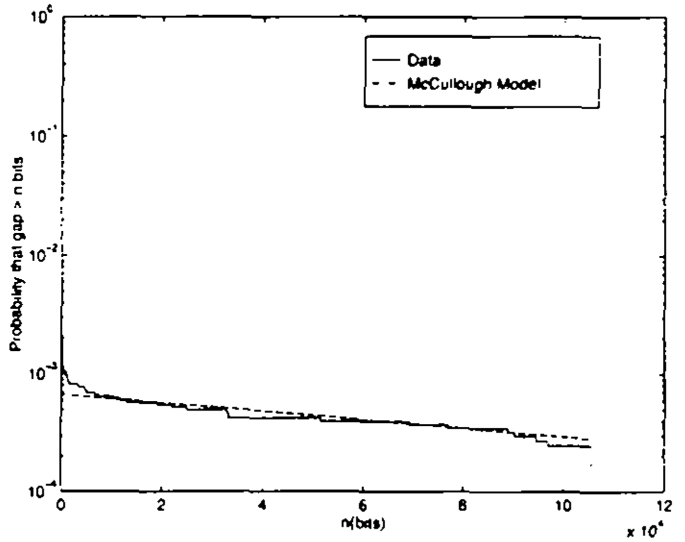

Figure 5.17: Error Gap Distribution Using Parameters Chosen to Match the Error Gap Distribution for Run 071016 for $\eta = -6dB$.

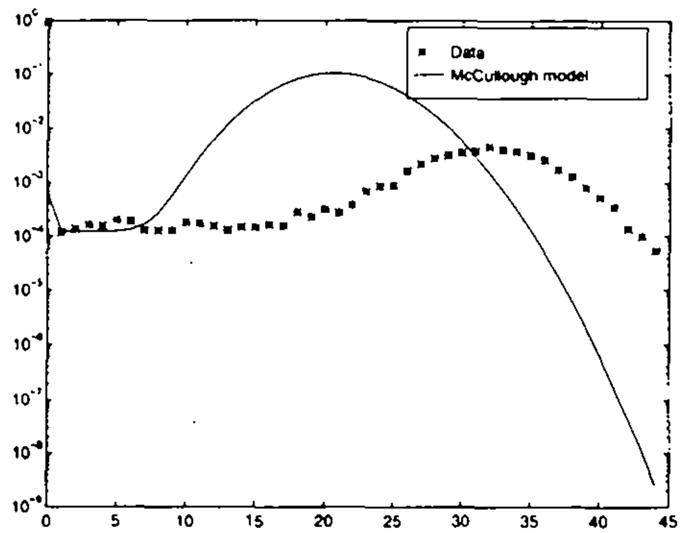

Figure 5.18: Block Error Probabilities Using Parameters Chosen to Match the Error Gap Distribution for Run 071016 for $\eta = -6dB$.

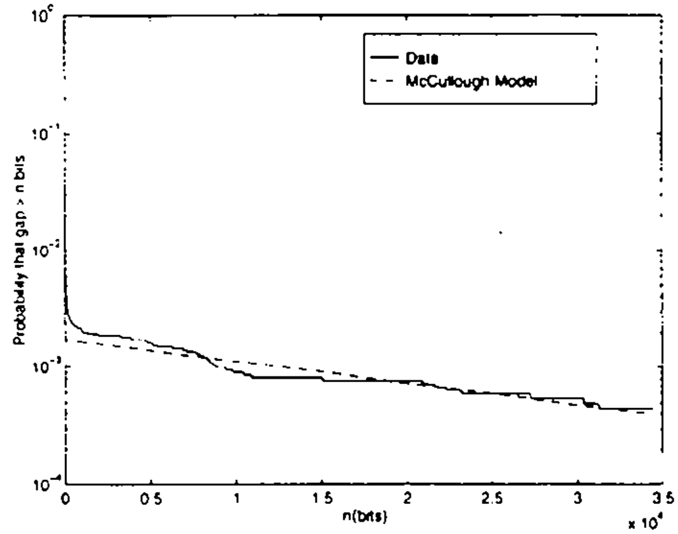

Figure 5.19: Error Gap Distribution Using Parameters Chosen to Match the Error Gap Distribution for Run 072406 for $\eta = -6dB$.

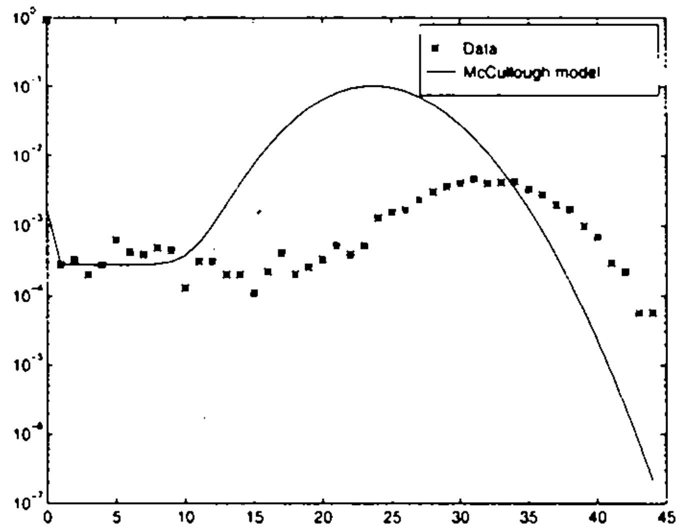

Figure 5.20: Block Error Probabilities Using Parameters Chosen to Match the Error Gap Distribution for Run 072406 for $\eta = -6dB$.

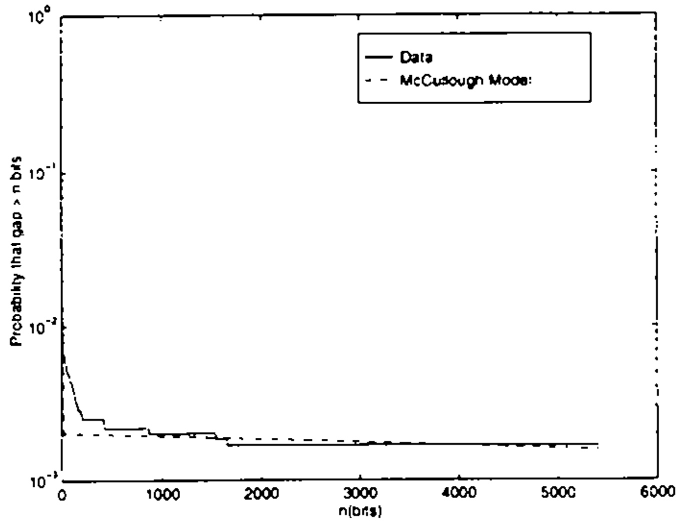

Figure 5.21: Error Gap Distribution Using Parameters Chosen to Match the Error Gap Distribution for Run 072411 for $\eta = -6dB$.

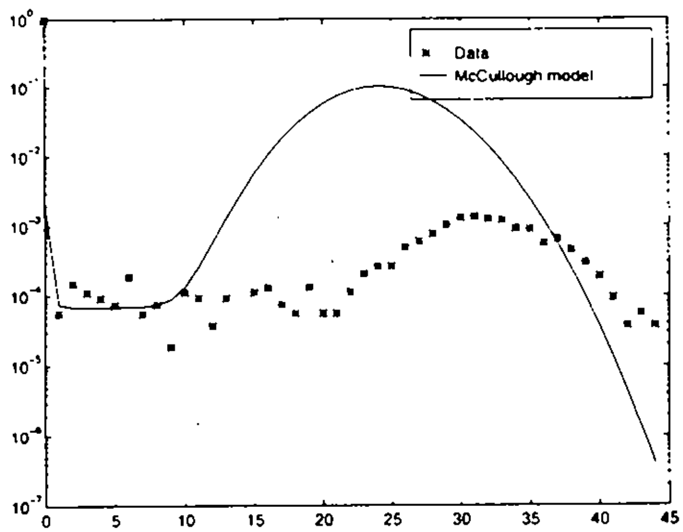

Figure 5.22: Block Error Probabilities Using Parameters Chosen to Match the Error Gap Distribution for Run 072411 for $\eta = -6dB$.

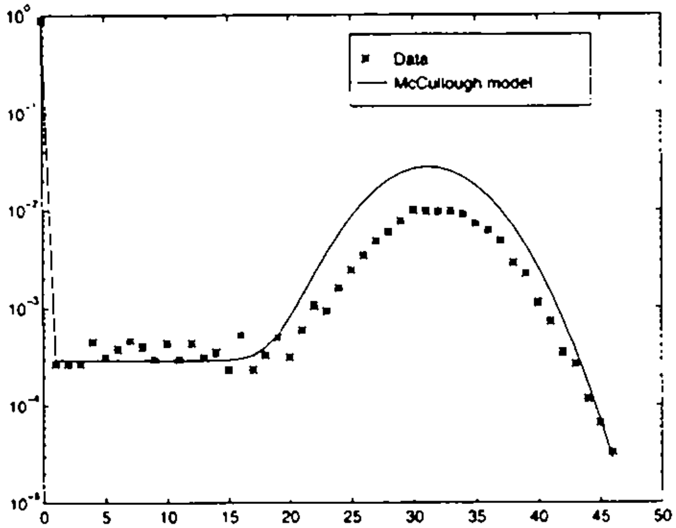

Figure 5.23: Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070906 for $\eta = -6dB$.

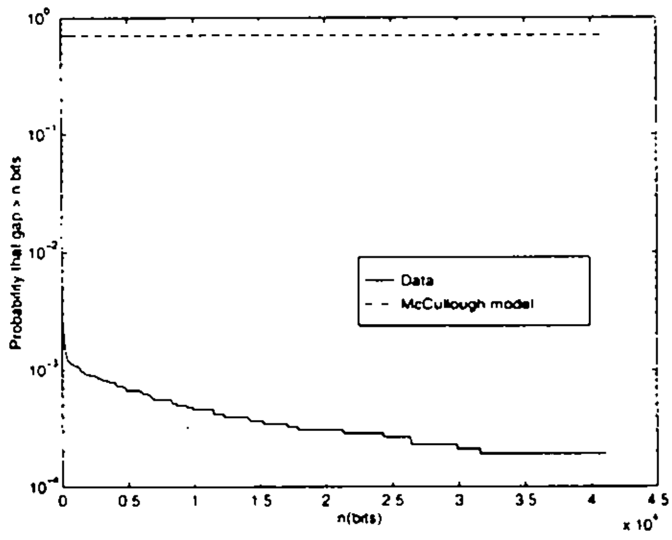

Figure 5.24: Error Gap Distribution with Parameters Chosen to Match the Block error Probabilities for Run 070906 for $\eta = -6dB$.

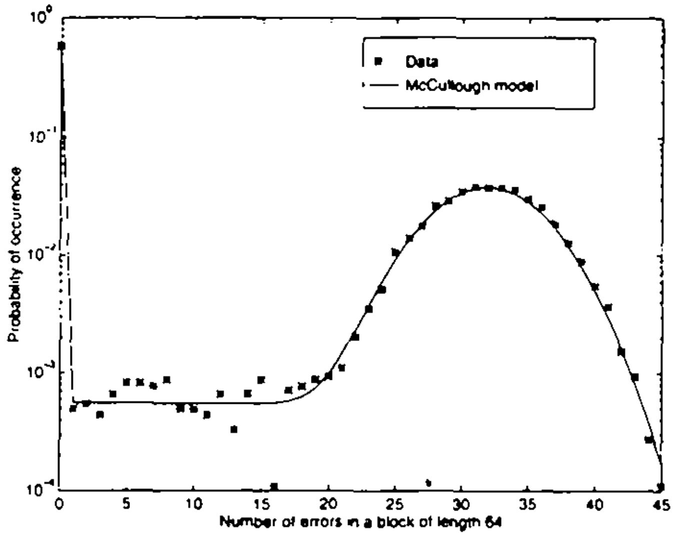

Figure 5.25: Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070912 for $\eta = -6dB$.

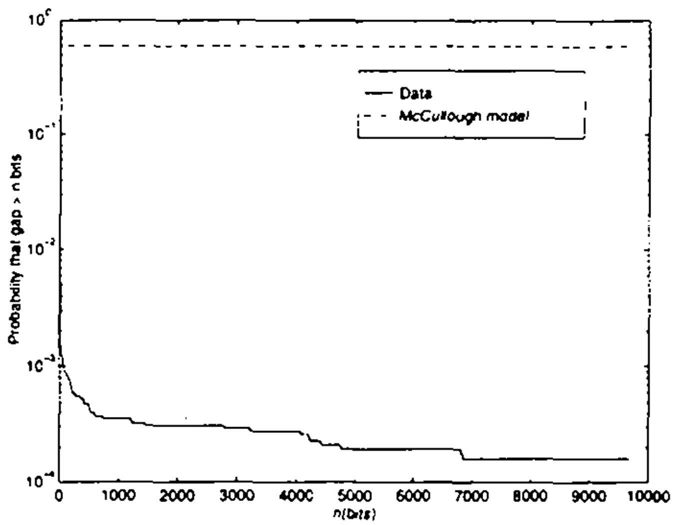

Figure 5.26: Error Gap Distribution with Parameters Chosen to Match the Block error Probabilities for Run 070912 for $\eta = -6dB$.

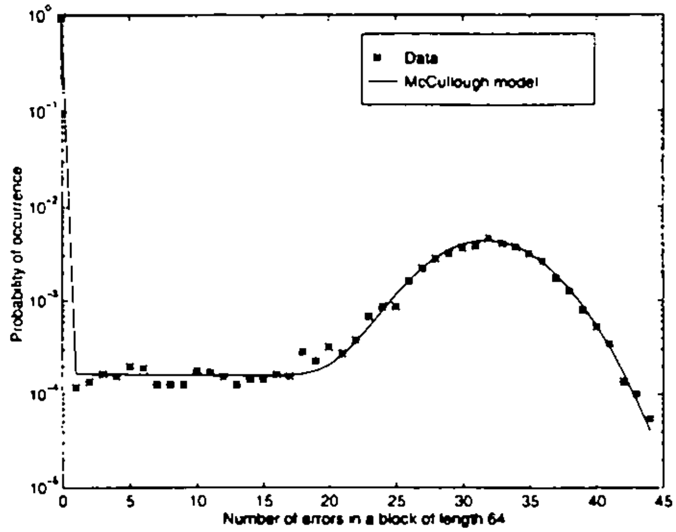

Figure 5.27: Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 071016 for $\eta = -6dB$.

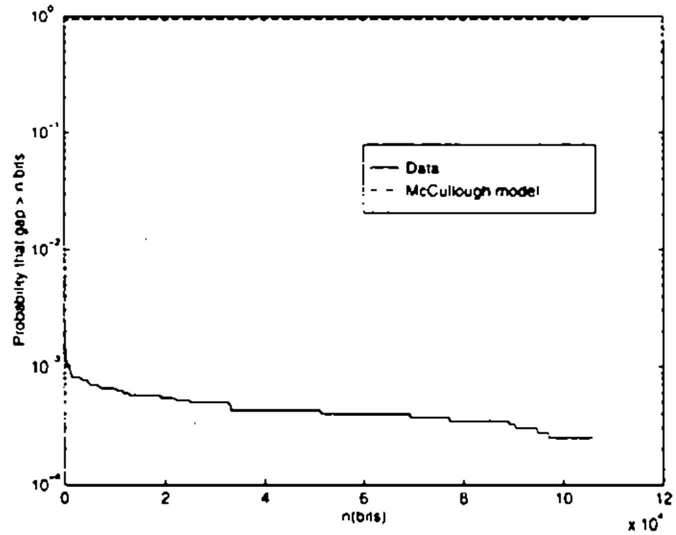

Figure 5.28: Error Gap Distribution with Parameters Chosen to Match the Block error Probabilities for Run 071016 for $\eta = -6dB$.

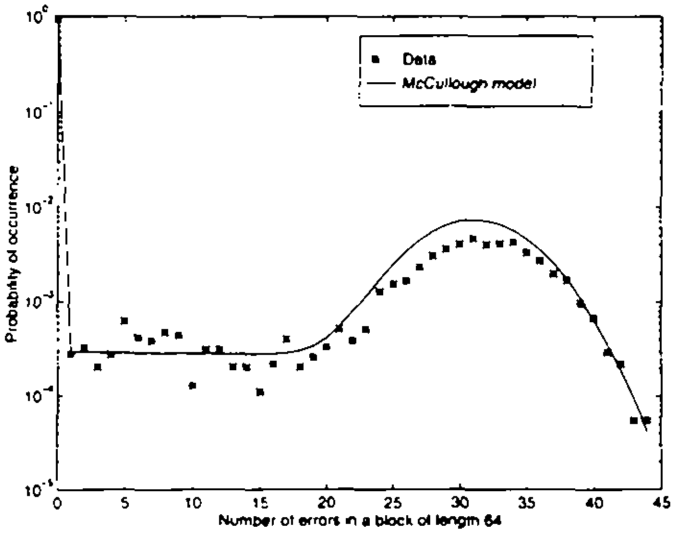

Figure 5.29: Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072406 for $\eta = -6dB$.

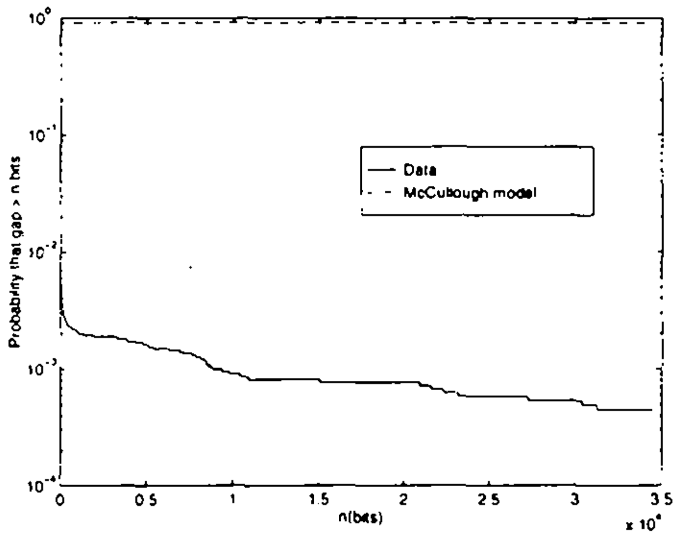

Figure 5.30: Error Gap Distribution with Parameters Chosen to Match the Block error Probabilities for Run 072406 for $\eta = -6dB$.

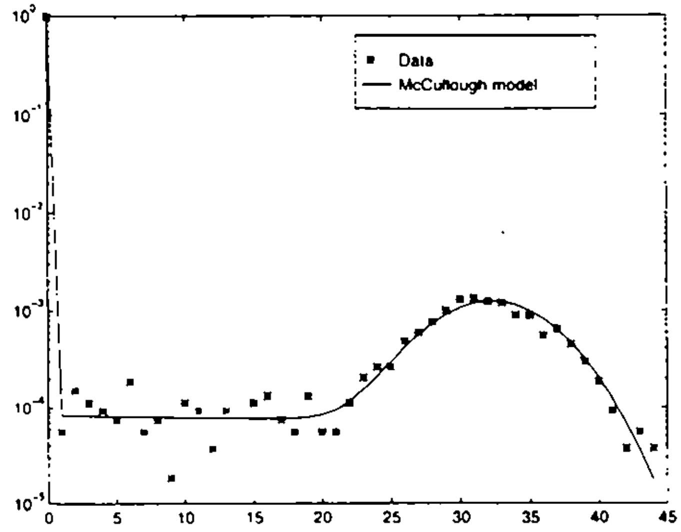

Figure 5.31: Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072411 for $\eta = -6dB$.

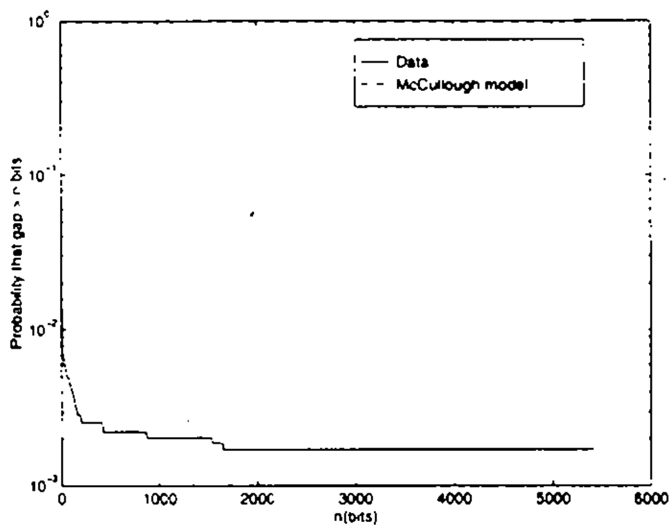

Figure 5.32: Error Gap Distribution with Parameters Chosen to Match the Block error Probabilities for Run 072411 for $\eta = -6dB$.

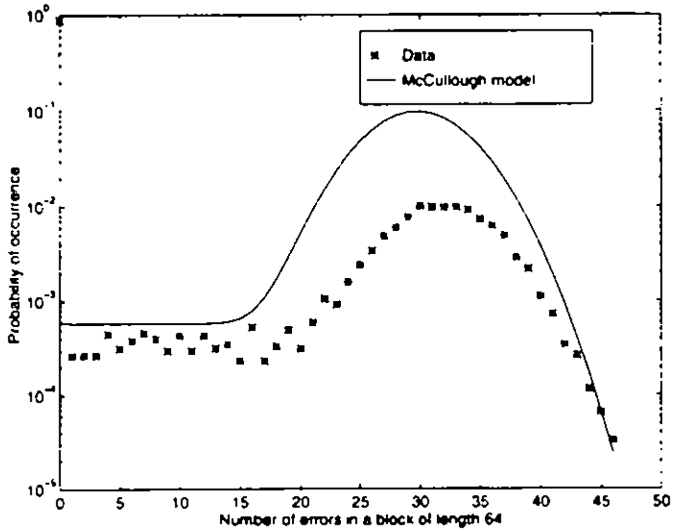

Figure 5.33: McCullough Model Block Error Probabilities for Run 070906 with $k = 430$ for $\eta = -6dB$.

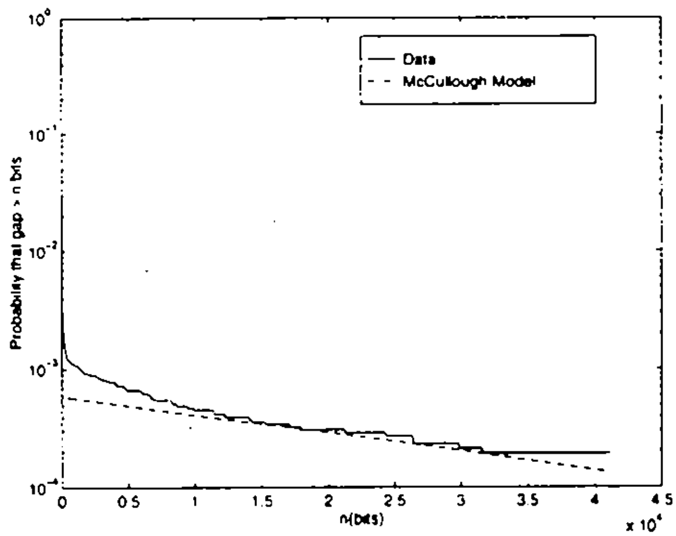

Figure 5.34: McCullough Model Error Gap Distribution for Run 070906 with $k = 430$ for $\eta = -6dB$.

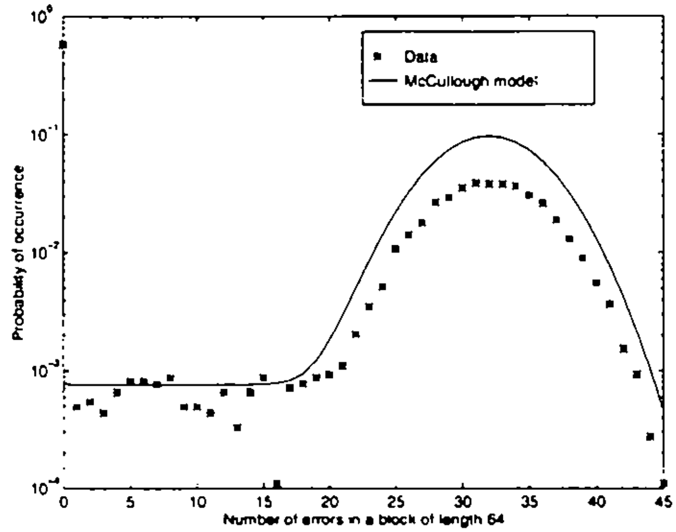

Figure 5.35: McCullough Model Block Error Probabilities for Run 070912 with $k = 20$ for $\eta = -6dB$.

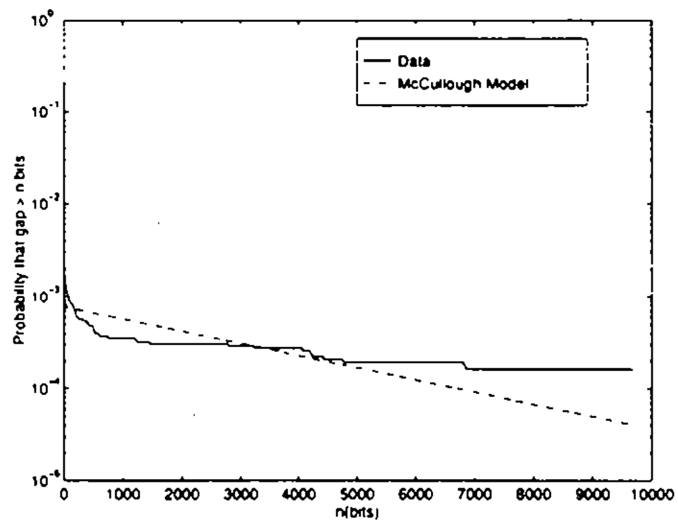

Figure 5.36: McCullough Model Error Gap Distribution for Run 070912 with $k = 20$ for $\eta = -6dB$.

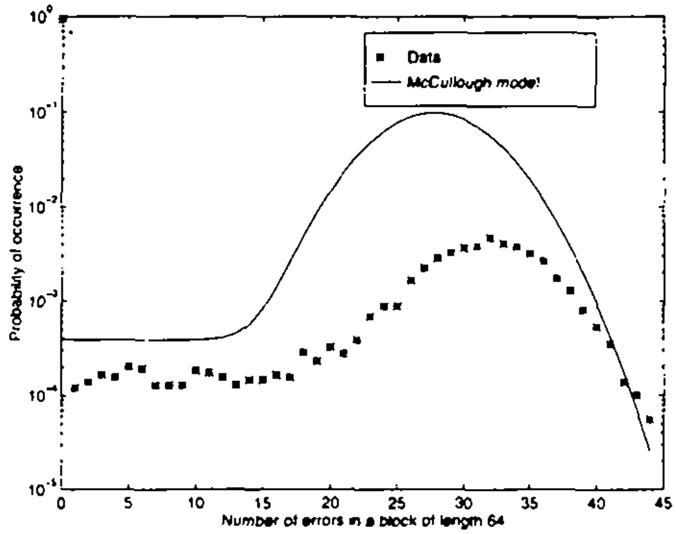

Figure 5.37: McCullough Model Block Error Probabilities for Run 071016 with $k = 1450$ for $\eta = -6dB$.

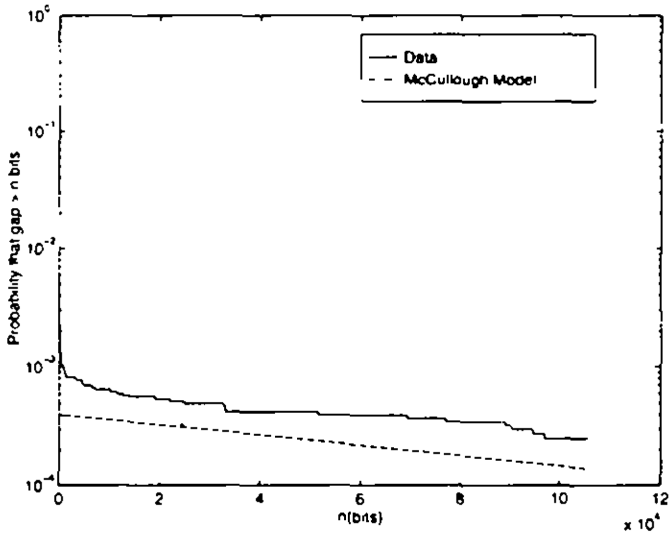

Figure 5.38: McCullough Model Error Gap Distribution for Run 071016 with $k = 1450$ for $\eta = -6dB$.

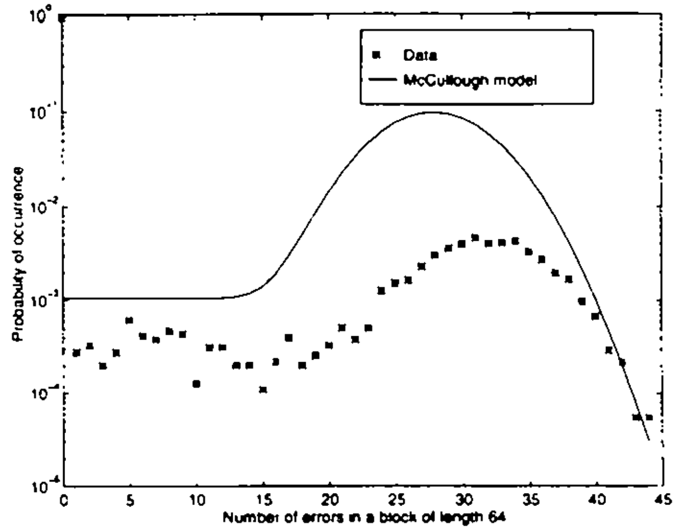

Figure 5.39: McCullough Model Block Error Probabilities for Run 072406 with $k = 740$ for $\eta = -6dB$.

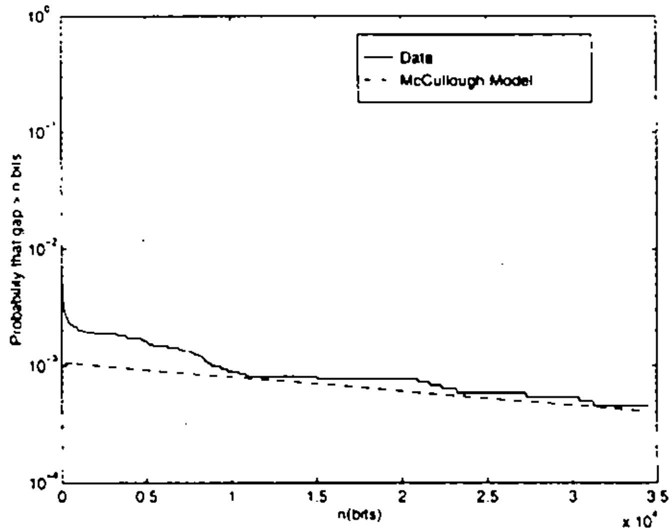

Figure 5.40: McCullough Model Error Gap Distribution for Run 072406 with $k = 740$ for $\eta = -6dB$.

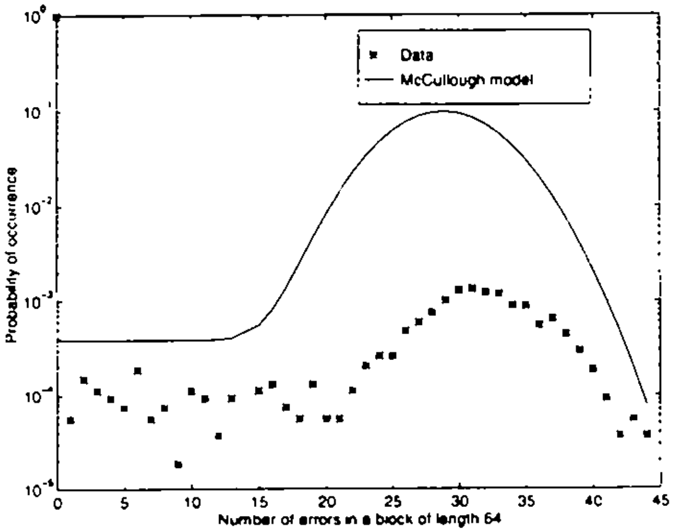

Figure 5.41: McCullough Model Block Error Probabilities for Run 072411 with $k = 1660$ for $\eta = -6dB$.

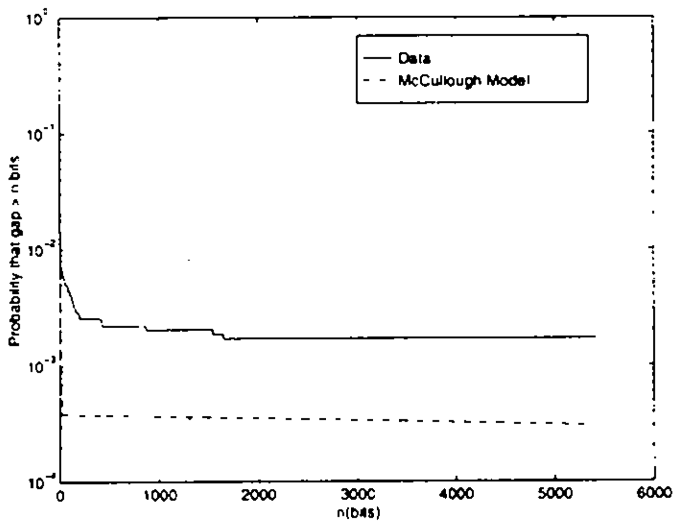

Figure 5.42: McCullough Model Error Gap Distribution for Run 072411 with $k = 1660$ for $\eta = -6dB$.

Chapter 6

THE FRITCHMAN MODEL

The Fritchman model [8] takes a different approach to modeling the data. In this model, the states represent error bit values instead of BSC's. In this way, the error bits are determined directly by the state sequence.

6.1 The Model

As shown in Figure 6.1, the N states of the Fritchman model are divided into two groups, A and B. Group A is a set of k error-free states, and group B is a group of $(N-k)$ error states. Let $Z = \{z_i : i = 1, 2, \cdots\}$ represent a sequence of states in which the transitions occur synchronously with the transmission of the input sequence $X = \{x_i : i = 1, 2, \cdots\}$. Next, let ϕ be a mapping from the states $S = \{1, 2, \cdots\}$ onto $\{0, 1\}$, where

$$\phi(i) = \begin{cases} 0 & \text{for } i \in A \\ 1 & \text{for } i \in B \end{cases} \tag{6.1}$$

Using this notation, the error sequence e_i can be represented by $\phi(z_i)$. This in turn represents the errors that occur on the channel.

In an attempt to simplify the model, transitions are allowed from error-free states to any error state, but not between the error-free states themselves (see Figure 6.1). This is assumed in order to make the model tractable.

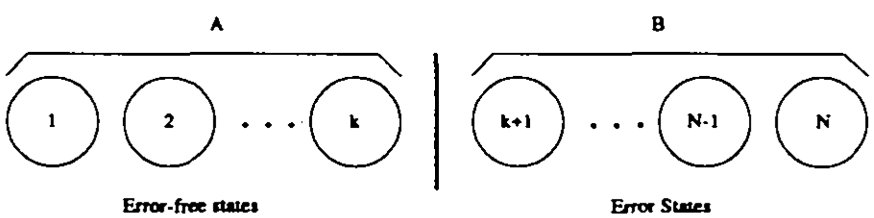

Figure 6.1: Partitioning the State Space.

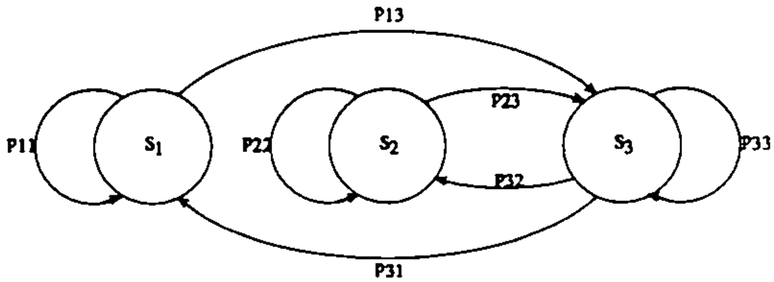

Figure 6.2: Simplified Fritchman Model with One Error State.

6.2 The Error Gap Distribution

The determination of the error gap distribution using Fritchman's finite state Markov model follows the procedure outlined in [8]. First, let the discrete collection of random variables $\{z_i : i = 1, 2, \cdots\}$ be a stationary Markov chain with state space $S = \{1, 2, \ldots, N\}$. The single-step and m-step transition probabilities are denoted by p_{ij} and $p_{ij}(m)$ respectively, and the corresponding $N \times N$ matrices as $P = [p_{ij}]$ and $P(m) = [p_{ij}(m)]$. Then for a stationary chain $P(m) = P^m$. Also, if $\{p_i : i = 1, 2, \cdots, N\}$ are the initial probabilities, for a strictly stationary chain, p_i can be determined from P using

$$p_i = \sum_{j=1}^{N} p_i p_{ji} \text{ for } i = 1, 2, \cdots, N \tag{6.2}$$

where

$$\sum_{i=1}^{N} p_i = 1. \tag{6.3}$$

If P is restricted to be similar to a diagonal matrix and $L^{(i)} = (l_1^{(i)}, \cdots, l_N^{(i)})$ and $R^{(i)} = (r_1^{(i)}, \cdots, r_N^{(i)})$ are the left and right eigenvectors of P corresponding to the eigenvalue λ_i, then the m-step transition probabilities can be expanded as

$$p_{ij}(m) = \sum_{\nu=1}^{N} C_\nu r_i^{(\nu)} l_i^{(\nu)} \lambda_\nu^m \tag{6.4}$$

where

$$C_\nu = \left[\sum_{i=1}^{N} r_i^{(\nu)} l_i^{(\nu)} \right]^{-1}. \tag{6.5}$$

Now, referring to Figure 6.1,

$$P(0^m|1) = \frac{P(10^m)}{P(1)}$$
$$= \frac{\sum_{i_0 \in B} \sum_{i_1 \in A} \cdots \sum_{i_m \in A} P(z_0 = i_0, z_1 = i_1, \cdots, z_m = i_m)}{\sum_{i_0 \in B} P(z_0 = i_0)}. \quad (6.6)$$

The numerator can be expressed as

$$\sum_{i_0 \in B} \sum_{i_1 \in A} P(z_0 = i_0) P(z_1 = i_1 | z_0 = i_0) \sum_{i_2 \in A} \cdots \sum_{i_m \in A} P(z_2 = i_2 \cdots z_m = i_m | z_1 = i_1) \quad (6.7)$$

Also, since $i_2, i_3, \cdots, i_k \in A$, the second part of the numerator summation, can be denoted as $P(0^{m-1}|z_1 = i_1)$ where

$$P(0^{m-1}|z_1 = i_1) = \begin{cases} \sum_{i_2 \in A} \cdots \sum_{i_m \in A} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{m-1} i_m} & \text{for } m \geq 2 \\ 1 & \text{for } m = 1. \end{cases} \quad (6.8)$$

While this gives an expression in terms of the basic model parameters, a more useful form is obtained by considering the $(m-1)$-step transition probability $p_{i_1 i_m}(m-1)$, which is the probability that, starting from state i_1, the process will be in state i_m $(m-1)$ transitions later. In going from i_1 to i_m the process can pass through any of the states of A and B. Therefore, $p_{i_1 i_m}(m-1)$ is the sum of the probabilities of all the state sequences that begin with state i_1 and end with state i_m after $m-1$ transitions; that is,

$$p_{i_1 i_m}(m-1) = \sum_{i_2 \in A \bigcup B} \sum_{i_3 \in A \bigcup B} \sum_{i_{m-1} \in A \bigcup B} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{m-1} i_m}$$
$$= \begin{cases} \sum_{i_2 \in A} \sum_{i_3 \in A} \cdots \sum_{i_{m-1} \in A} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{m-1} i_m} + \Pr(S_b) & \text{for } m \geq 2 \\ 1 & \text{for } m = 1 \end{cases} \quad (6.9)$$

where $\Pr(S_b)$ represents the sums over all sequences involving states of B. In this expression, the first term is the probability of moving from state i_1 to i_m in $m-1$ transitions without passing through any of the states of B. This is denoted as ${}_B p_{i_1 i_m}(m-1)$. From (6.8) and (6.9), it is seen that

$$P(0^{m-1}|z_1 = i_1) = \begin{cases} \sum_{i_m \in A} {}_B p_{i_1 i_m}(m-1) & \text{for } m \geq 2 \\ 1 & \text{for } m = 1. \end{cases} \quad (6.10)$$

So, an expression for $P(0^{m-1}|z_1 = i_1)$ can be obtained from ${}_Bp_{i_1 i_m}(m-1)$, which in turn can be obtained from the $(m-1)$ step transition probability by subtracting all the terms that involve transitions through states of B.

Instead of doing the subtraction directly, the same result can be obtained by modifying the model. In the modified model, the process is characterized by the transition matrix $\widehat{P}$, where

$$\widehat{P} = \begin{bmatrix} P_A & P_{AB} \\ 0 & I \end{bmatrix}. \tag{6.11}$$

P_A represents transitions among the states of A and P_{AB} transitions from the states of A to B. I, of course, is an identity matrix. In the original and the modified process, P_A and P_{AB} are the same. However, in the modified process, transitions from states B to A are zero, i.e., an absorbing process. As a result of this, $p_{i_1,i_m}(m-1)$ is the same for both models when $i_1, i_m \in A$. Also, for the modified process, since the probability of a transition from A to B to A or from B to A is zero, $\widehat{p}_{i_1 i_m}(m-1) =_B p_{i_1 i_m}(m-1)$ when $i_1, i_m \in A$. But, by definition of $\widehat{P}(m-1)$,

$$\begin{aligned} \widehat{P}(m-1) &= [\widehat{p}_{i_1 i_m}(m-1)] \\ &= \widehat{P}^{m-1} \\ &= \begin{cases} \begin{bmatrix} P_A^{m-1} & 0 \\ 0 & I \end{bmatrix} & \text{for } m \geq 2 \\ = I & \text{for } m = 1 \end{cases} \end{aligned} \tag{6.12}$$

where the elements of P_A^{m-1} are the $(m-1)$ step transition probabilities $\widehat{p}_{i_1 i_m}(m-1)$ when $i_1 i_m \in A$. The probabilities ${}_Bp_{i_1 i_m}(m-1)$ for $i_1 i_m \in A$ are the elements of P_A^{m-1}. As a result of (6.4), the elements of P_A^{m-1} can be written in terms of the eigenvalues $\widehat{\lambda}_i$ and left and right eigenvectors $\widehat{L}^{(i)}$ and $\widehat{R}^{(i)}$ of the transition matrix P_A if P_A is similar to a diagonal matrix. The expression so obtained is given by

$${}_Bp_{i_1 i_m}(m-1) = \begin{cases} \sum_{\nu \in A} \widehat{C}_\nu \widehat{r}_{i_1}^{(\nu)} \widehat{l}_{i_m}^{(\nu)} \widehat{\lambda}_\nu^{m-1} & \text{for } m \geq 2 \\ = 1 & \text{for } m = 1 \end{cases} \tag{6.13}$$

where

$$\widehat{C}_\nu = [\sum_{s=1}^{k} \widehat{r}_s^{(\nu)} \widehat{l}_s^{(\nu)}]^{-1}.$$

From (6.10),

$$P(0^{m-1}|z_1 = i_1) = \begin{cases} \sum_{i_m \in A} \sum_{\nu \in A} \widehat{C}_\nu \widehat{r}_{i_1}^{(\nu)} \widehat{l}_{i_m}^{(\nu)} \widehat{\lambda}_\nu^{m-1} & \text{for } i_1 \in A, \text{ m } \geq 2 \\ = 1 & \text{for } m = 1 \end{cases} \tag{6.14}$$

and from (6.6),

$$P(0^m|1) = \sum_{\nu=1}^{k} \widehat{f}(\nu) \widehat{\lambda}_\nu^m \text{ for } m \geq 1 \tag{6.15}$$

where

$$\widehat{f}(\nu) = \begin{cases} \left(\frac{\widehat{C}_\nu}{\widehat{\lambda}_\nu}\right) \frac{\sum_{i=k+1}^{N} \sum_{j=1}^{k} \sum s=1^k p_i p_{ij} \widehat{r}_j^{(\nu)} \widehat{l}_s^{(\nu)}}{\sum_{i=k+1}^{N} p_i} & \text{for } m \geq 2 \\ \frac{1}{\widehat{\lambda}_\nu} \frac{\sum_{i=k+1}^{N} \sum_{j=1}^{k} p_i p_{ij}}{\sum_{i=k+1}^{N} p_i} & \text{for } m = 1. \end{cases} \tag{6.16}$$

This shows that the general form of the error gap distribution for a function of an N-state Markov chain with k error-free states is the weighted sum of at most k exponentials.

Similarly, for the error cluster distribution

$$P(1^m|0) = \sum_{\nu=1}^{N-k} \overline{f}(\nu) \overline{\lambda}_\nu^{m-1} \tag{6.17}$$

if

$$\overline{f}(\nu) = \left(\frac{\overline{C}_\nu}{\overline{\lambda}_\nu}\right) \frac{\sum_{i=1}^{k} \sum_{j=k+1}^{N} \sum s = k+1^N p_i p_{ij} \overline{r}_j^{(\nu)} \overline{l}_s^{(\nu)}}{\sum_{i=k+1}^{N} p_i} \tag{6.18}$$

and

$$\overline{C}_\nu = [\sum_{s=1}^{N-k} \overline{r}_s^{(\nu)} \overline{l}_s^{(\nu)}]^{-1}. \tag{6.19}$$

In this case, $\overline{\lambda}_\nu$ are the eigenvalues of P_B, the matrix representing transitions among the error states. Here, $\overline{r}_i^{(\nu)}$ and $\overline{l}_i^{(\nu)}$ are the corresponding elements of the right and left eigenvectors. Thus, the error-cluster distribution is also the weighted sum of exponentials.

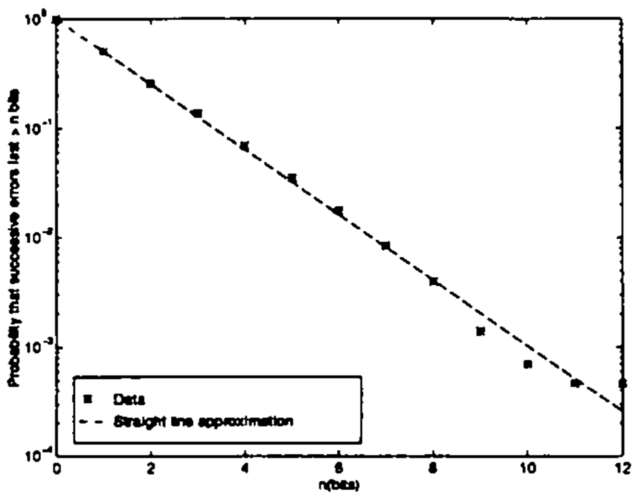

Figure 6.3: The Error-cluster Distribution.

6.3 Block Error Probabilities

A plot of a typical error-cluster distribution for the data is shown in Figure 6.3. The plot is very well approximated by a straight line. From the discussion in the previous section, this implies that there is only a single error state in the model(one weighted exponential). The expression for the error gap distribution is obtained using (6.15) and (6.16). Since P_A is diagonal, the eigenvalues $\overline{\lambda}_i$ of P_A are the diagonal elements $p_{11}p_{22}\cdots p_{N-1N-1}$. Also, since P_A is symmetric, the corresponding right and left eigenvectors are equal, and, for a diagonal matrix, are

$$\widehat{R}_i^{(i)} = \widehat{L}_i^{(i)} = (\widehat{r}_k^{(i)} = \widehat{l}_k^{(i)} = \delta_{ki} : k - 1, \cdots, N-1) \tag{6.20}$$

for $i = 1, \cdots, N-1$, and

$$\widehat{C}_i = 1 \text{ for } i = 1, \cdots, N-1. \tag{6.21}$$

From (6.16),

$$
\begin{aligned}
p_{\nu\nu}\widehat{f}(\nu) &= \sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \widehat{r}_i^{(\nu)}\widehat{l}_i^{(\nu)} p_{Nk} \\
&= \sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \delta_{j\nu}\delta_{k\nu} p_{Nk} \\
&= p_{N\nu}
\end{aligned}
\tag{6.22}
$$

and from (6.15),

$$P(0^m|1) = \sum_{\nu=1}^{N-1} \frac{p_{N\nu}}{p_{\nu\nu}}(p_{\nu\nu})^m \text{ for } m \geq 1, \tag{6.23}$$

which is simply the error gap distribution $u(n)$ by letting $n = m$.

The result of (6.23) is that for the special case of single-error-state models represented in Figure 6.1, the error gap distribution uniquely specifies the model [8]. Appropriate transition probabilities can be found to find the best fit to the $u(n)$ curve.

Once the transition probabilities are found, the $P(m,n)$ distribution is derived in [14] using the recurrence relation

$$f_i(m,n) = \sum_{j=1}^{k} P_{ij} f_j(m, n-1) + \sum_{j=k+1}^{N} P_{ij} f_j(m-1, n-1) \tag{6.24}$$

where i is the state, and $f_i(m,n)$ is the probability of exactly m transitions into the error states(in our case, there is only one) in n steps, given that the initial state is i. Since the initial state could be any one of the states, all possibilities can be accounted for by setting

$$P(m,n) = \sum_{i=1}^{N} p_i f_i(m,n). \tag{6.25}$$

The value of p_i is the steady state probability of state i. Also, in evaluating (6.24), the initial conditions are

$$
\begin{aligned}
f_j(m,n) &= 0, \text{ for } m > n \\
f_j(m,n) &= 0, \text{ for } n < 0 \text{ or } m < 0 \\
f_j(0,0) &= 1.
\end{aligned}
$$

6.4 Comparison of the Fritchman Model and the Data

The Fritchman model is evaluated using the following methods.

1. Find the values of the transition probability matrix that best fit the error gap distribution curve. From these values, derive the block error probabilities.

2. Find the values of the transition probability matrix that best fit the block error probabilities. From these values, compare the error gap distribution of the data with the model.

The closest fit to the error gap distribution results in

$$P_{070906} = \begin{bmatrix} 0.999792 & 0 & 0.000208 \\ 0 & 0.999976 & 0.000024 \\ 0.000876 & 0.000447 & 0.998677 \end{bmatrix} \tag{6.26}$$

$$P_{070912} = \begin{bmatrix} 0.993361 & 0 & 0.006639 \\ 0 & 0.999869 & 0.000131 \\ 0.000924 & 0.000446 & 0.998630 \end{bmatrix} \tag{6.27}$$

$$P_{071016} = \begin{bmatrix} 0.964321 & 0 & 0.035679 \\ 0 & 0.999991 & 0.000009 \\ 0.000717 & 0.000685 & 0.998598 \end{bmatrix} \tag{6.28}$$

$$P_{072406} = \begin{bmatrix} 0.999999 & 0 & 0.000001 \\ 0 & 0.999922 & 0.000078 \\ 0.000350 & 0.001574 & 0.998076 \end{bmatrix} \tag{6.29}$$

and

$$P_{072411} = \begin{bmatrix} 0.621006 & 0 & 0.378994 \\ 0 & 0.999944 & 0.000056 \\ 0.008127 & 0.002113 & 0.989760 \end{bmatrix} \tag{6.30}$$

. The row in each matrix gives the current state, and the column in each matrix gives the next state. The value at each point is simply the probability of moving from the current state to the next state.

In Figures 6.4, 6.6, 6.8, 6.10, and 6.12 the plots of the error gap distribution from both the data and the model are presented.

Unlike the earlier models in which the model for the EGD did not perform well for smaller values of n, the Fritchman model provides a closer approximation to the smaller and larger values of n for the two category III runs. However, for the category II runs, there is only mild improvement for the smaller values of n. The transition probability matrices for these models are

Using these parameters to solve for $P(m,n)$, the results are shown in Figures 6.5, 6.7, 6.9, 6.11, and 6.13.

The results show that the model values of $P(m,n)$ are smaller than the given data values, and the "hump" in the data is not approximated by the model at all. Yet the model still provides better fits to $P(m,n)$ than the Gilbert or Elliott models given the error gap distribution.

Finding a closest fit to the values of $\overline{P}(m,n)$ yields transition probabilities given by

$$P_{070906} = \begin{bmatrix} 0.977238 & 0 & 0.022762 \\ 0 & 0.999504 & 0.000496 \\ 0.000760 & 0.000297 & 0.998943 \end{bmatrix} \tag{6.31}$$

$$P_{070912} = \begin{bmatrix} 0.967918 & 0 & 0.032082 \\ 0 & 0.999943 & 0.000057 \\ 0.003725 & 0.000013 & 0.996262 \end{bmatrix} \tag{6.32}$$

$$P_{071016} = \begin{bmatrix} 0.970380 & 0 & 0.029620 \\ 0 & 0.999767 & 0.000233 \\ 0.000492 & 0.002183 & 0.997325 \end{bmatrix} \tag{6.33}$$

$$P_{072406} = \begin{bmatrix} 0.968181 & 0 & 0.031819 \\ 0 & 0.999862 & 0.000138 \\ 0.001882 & 0.000336 & 0.997782 \end{bmatrix} \tag{6.34}$$

$$P_{072411} = \begin{bmatrix} 0.785913 & 0 & 0.214087 \\ 0 & 0.999866 & 0.000144 \\ 3.0 \times 10^{-9} & 0.000257 & 0.999743 \end{bmatrix}. \tag{6.35}$$

The results are shown in Figures 6.14, 6.16, 6.18, 6.20, and 6.22.

Again, the Fritchman model is unable to model the hump in the data where the number of errors in the block approaches around twenty-five and greater.

Looking at the resulting error gap distributions that come from these transition probability matrices are shown in Figures 6.15, 6.17, 6.19, 6.21, and 6.23.

The two curves, while similar in form, do not match the data. Again, the property of being able to model $u(n)$ or $\overline{P}(m,n)$ separately, but not simultaneously, appears. The biggest weakness with the Fritchman model is the inability to model the hump in the data where the number of errors in the block approaches twenty-five or so. This is due to the restrictions placed on the transition probabilities.

In an attempt to see if the restrictions on the model, such as no transitions being allowed between error-free states, are the cause of the difficulty in matching both curves simultaneously, or model the hump, a trial was done with the five sample runs allowing transitions from any state to any other state. In other words, the model is now fully connected. The results in finding $P(m,n)$ along with the corresponding $u(n)$ plots are shown in Figures 6.24 through 6.33. The transition matrices for these curves are given by

$$P_{070906} = \begin{bmatrix} 0.999904 & 0.000047 & 0.000048 \\ 0.000813 & 0.499879 & 0.499307 \\ 0.000833 & 0.496596 & 0.502570 \end{bmatrix} \tag{6.36}$$

$$P_{070912} = \begin{bmatrix} 0.999769 & 000116 & 0.000113 \\ 0.000349 & 0.500826 & 0.498825 \\ 0.000353 & 0.502543 & 0.497103 \end{bmatrix} \tag{6.37}$$

$$P_{071016} = \begin{bmatrix} 0.999961 & .000020 & 0.000019 \\ 0.000685 & 0.519654 & 0.479661 \\ 0.000666 & 0.519653 & 0.479680 \end{bmatrix} \tag{6.38}$$

$$P_{072406} = \begin{bmatrix} 0.999927 & 0.000038 & 0.000035 \\ 0.001400 & 0.501972 & 0.496627 \\ 0.001406 & 0.501297 & 0.497296 \end{bmatrix} \tag{6.39}$$

$$P_{072411} = \begin{bmatrix} 0.999982 & 0.0000090 & 0.000008 \\ 0.001027 & 0.521480 & 0.477492 \\ 0.000996 & 0.521480 & 0.477524 \end{bmatrix} \tag{6.40}$$

As can be seen from these plots, loosening up the restrictions on the state transitions provides a significant improvement in the modeling of the block error probabilities, but the error gap distribution curves are now not matched as well. Therefore, it can be concluded that while the Fritchman model can model the error gap distribution alone with more accuracy than previous models, it cannot model the block error probabilities unless the model is fully connected. And if it is, then it will not be able to model the error gap distribution as well. This characteristic has been present throughout all the models presented.

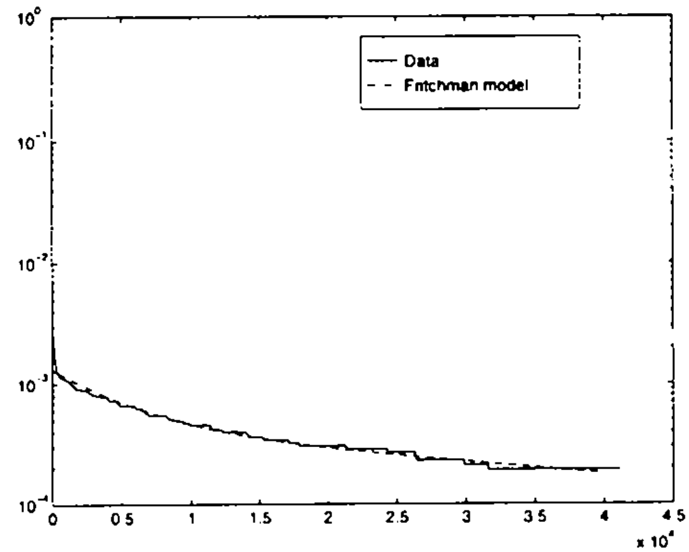

Figure 6.4: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Error Gap Distribution for Run 070906.

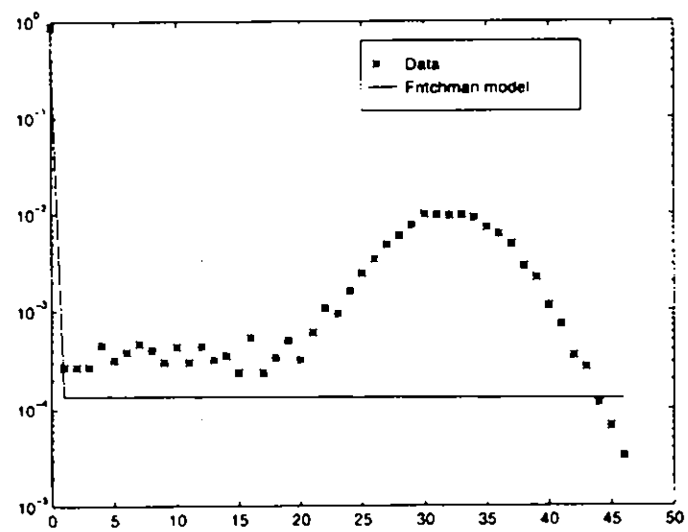

Figure 6.5: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Error Gap Distribution for Run 070906.

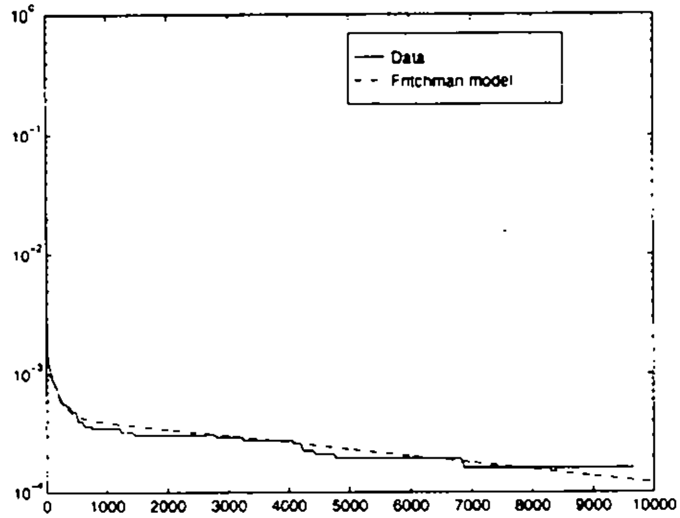

Figure 6.6: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Error Gap Distribution for Run 070912.

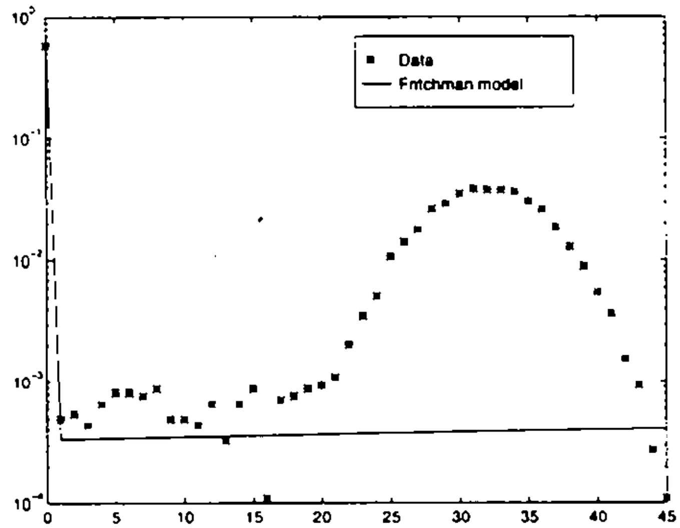

Figure 6.7: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Error Gap Distribution for Run 070912.

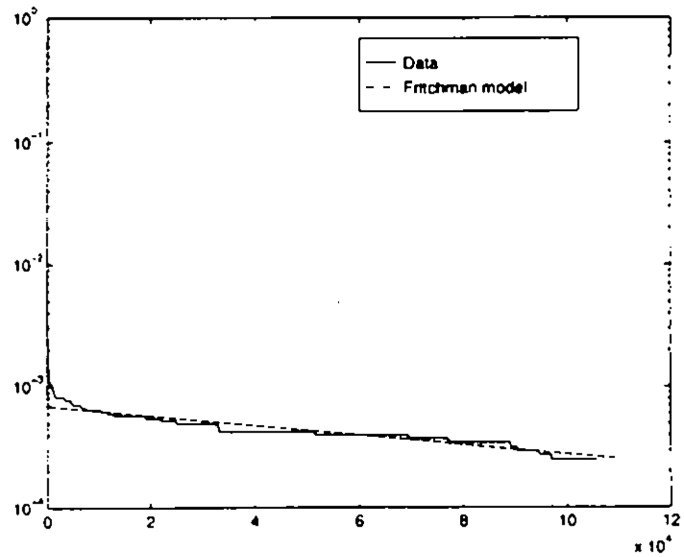

Figure 6.8: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Error Gap Distribution for Run 071016.

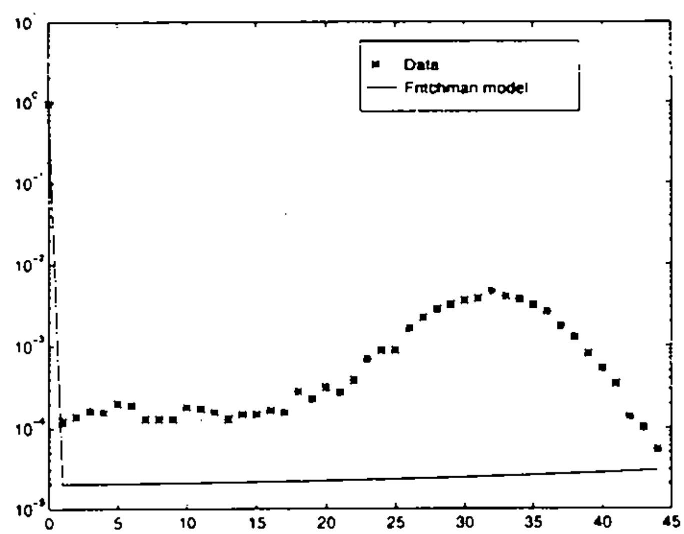

Figure 6.9: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Error Gap Distribution for Run 071016.

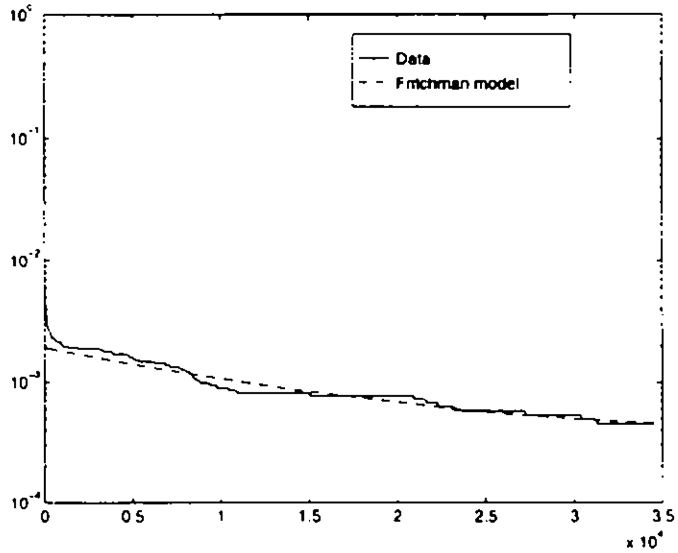

Figure 6.10: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Error Gap Distribution for Run 072406.

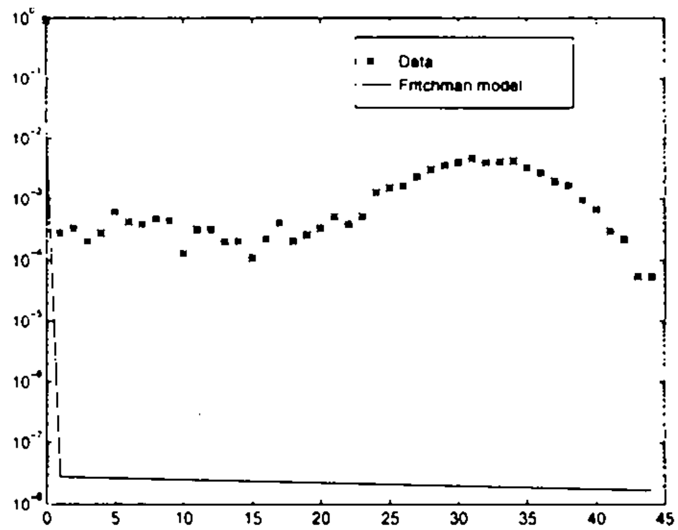

Figure 6.11: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Error Gap Distribution for Run 072406.

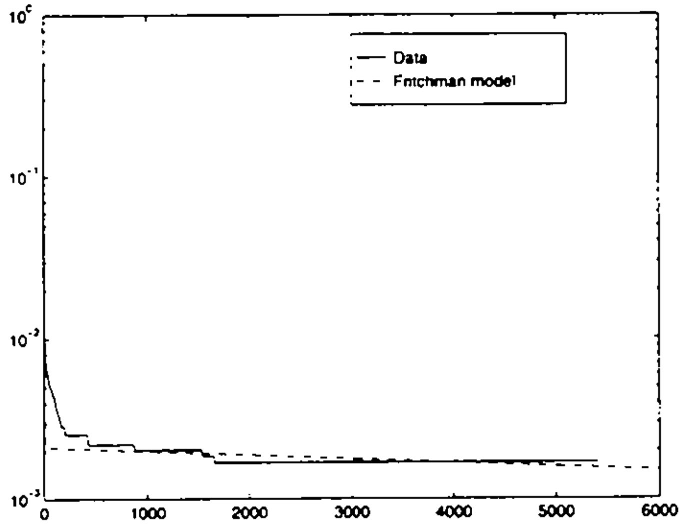

Figure 6.12: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Error Gap Distribution for Run 072411.

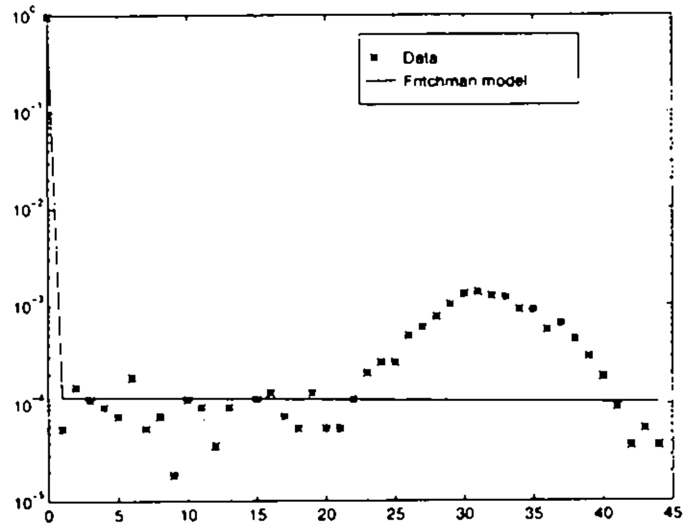

Figure 6.13: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Error Gap Distribution for Run 072411.

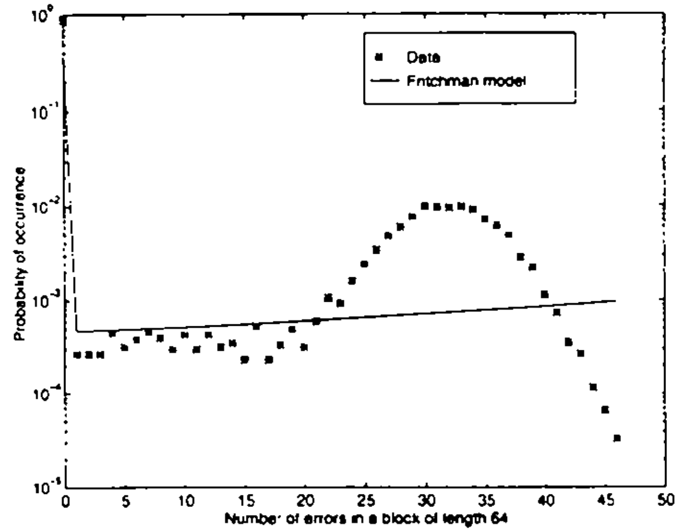

Figure 6.14: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070906 for $\eta = -6$ dB.

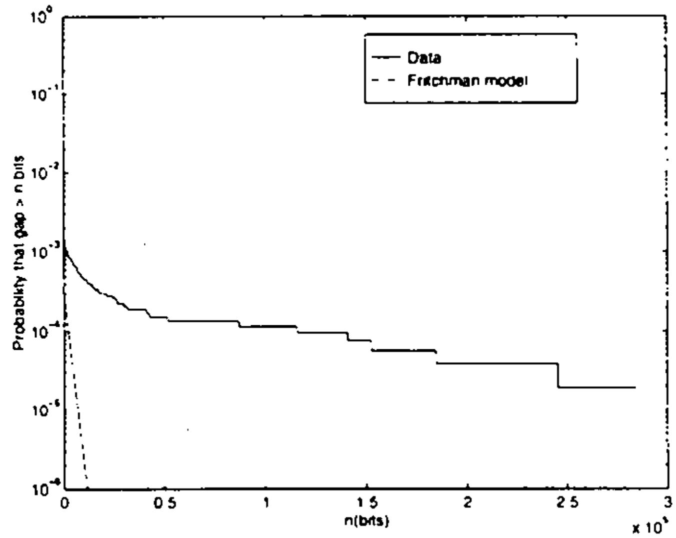

Figure 6.15: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 070906 for $\eta = -6$ dB.

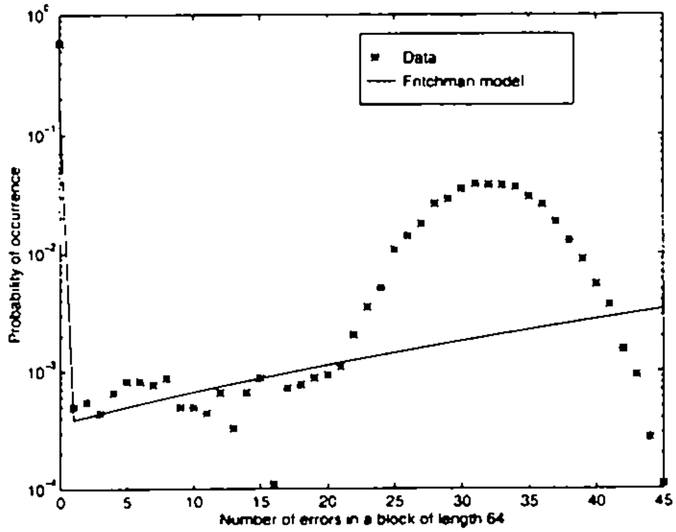

Figure 6.16: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070912 for $\eta = -6$ dB.

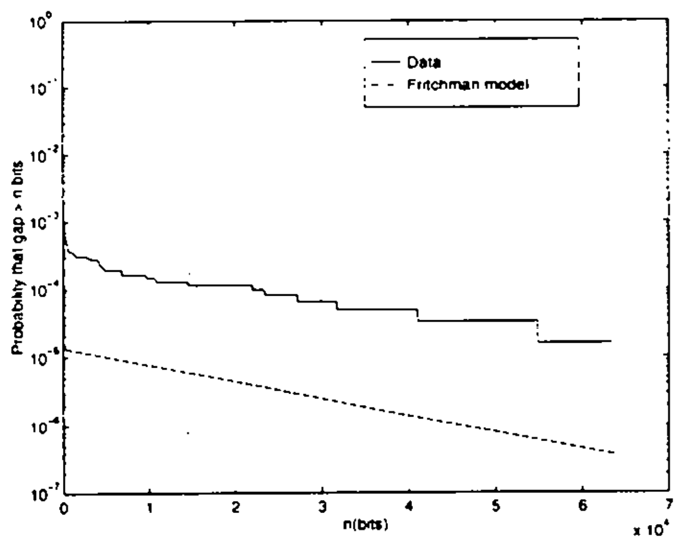

Figure 6.17: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 070912 for $\eta = -6$ dB.

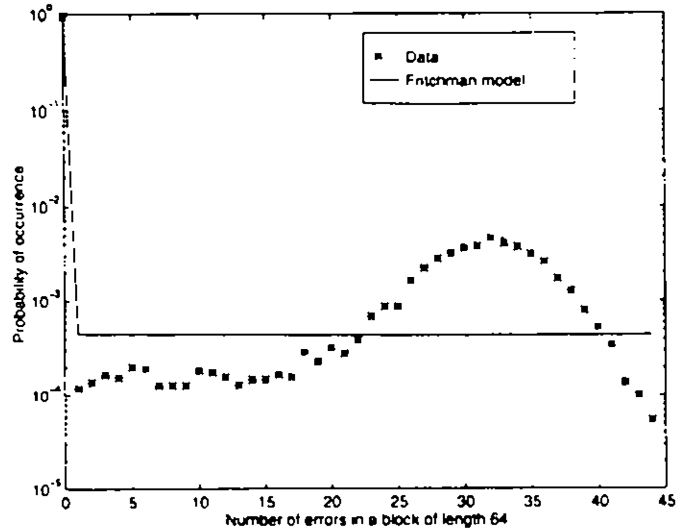

Figure 6.18: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 071016 for $\eta = -6$ dB.

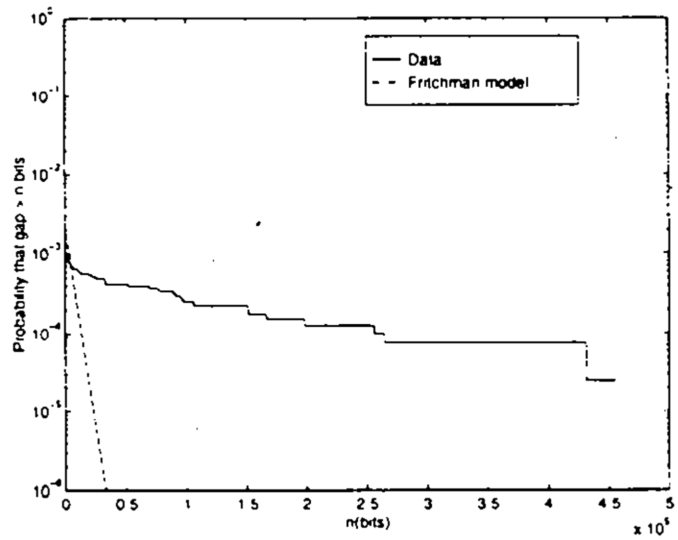

Figure 6.19: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 071016 for $\eta = -6$ dB.

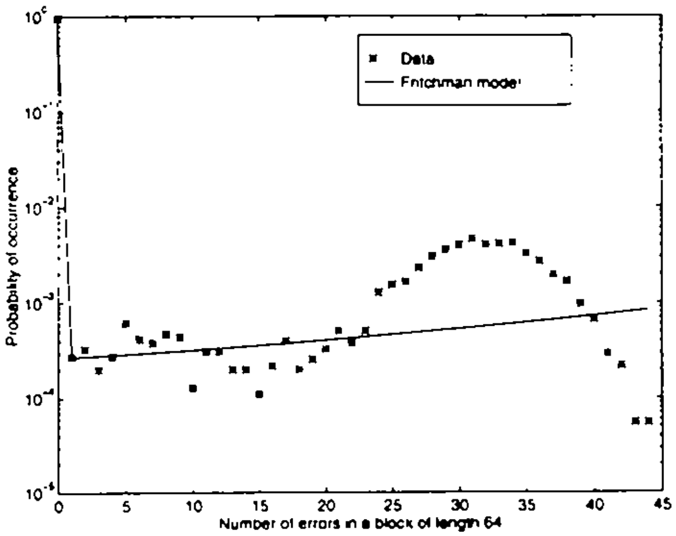

Figure 6.20: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072406 for $\eta = -6$ dB.

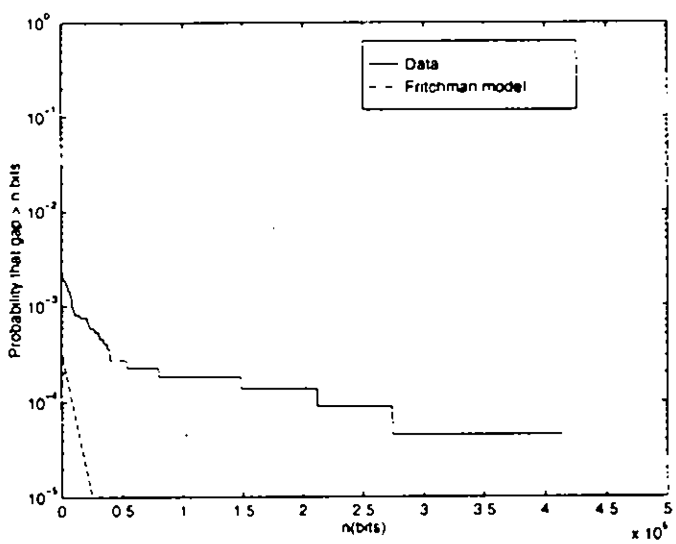

Figure 6.21: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 072406 for $\eta = -6$ dB.

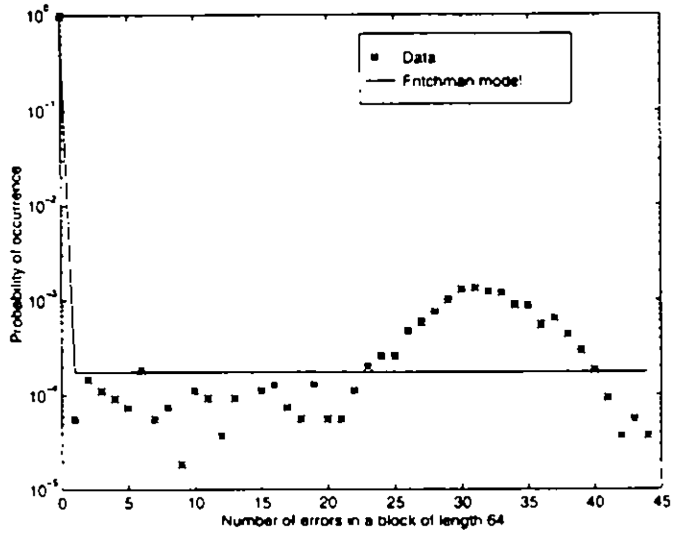

Figure 6.22: Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072411 for $\eta = -6$ dB.

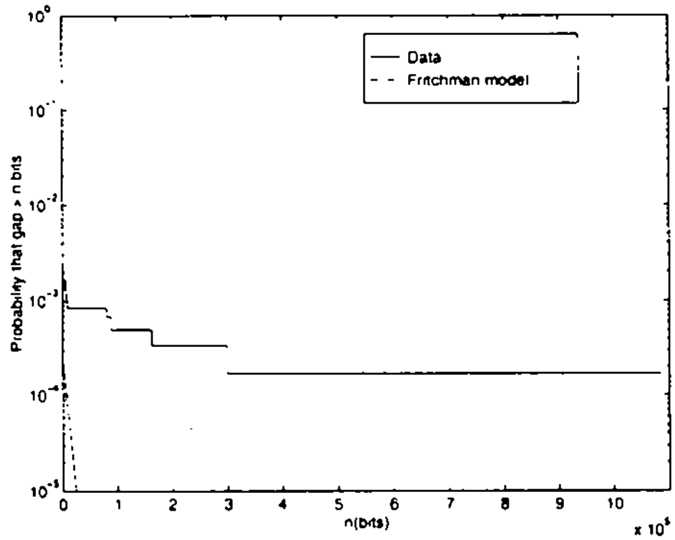

Figure 6.23: Fritchman Model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 072411 for $\eta = -6$ dB.

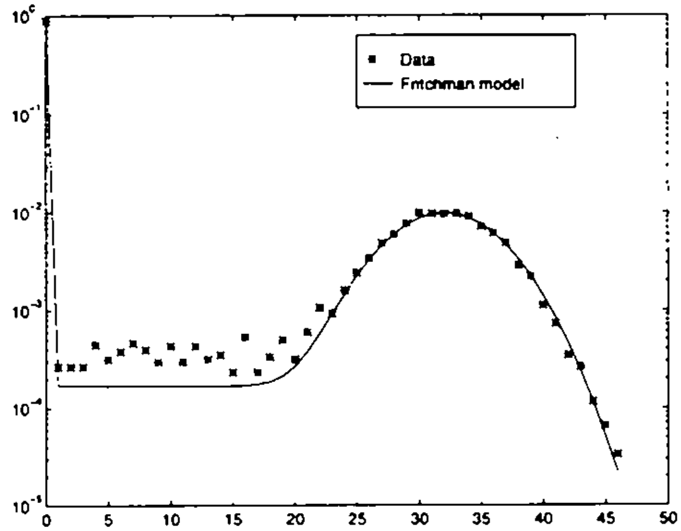

Figure 6.24: Fully Connected Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070906.

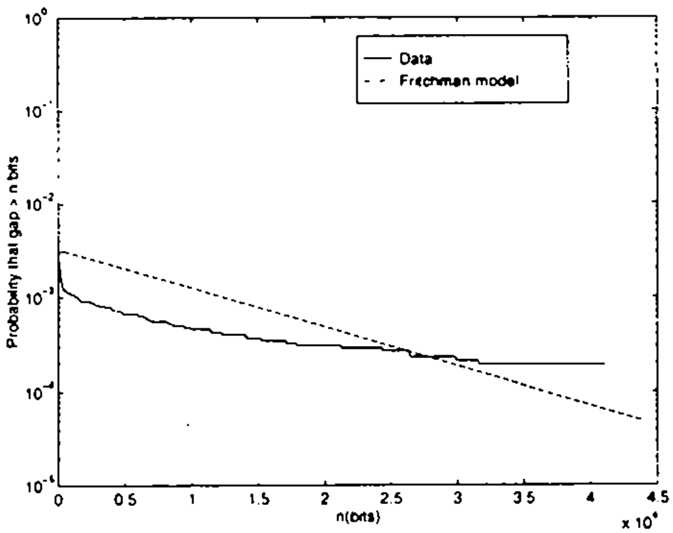

Figure 6.25: Fully Connected Fritchman model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 070906.

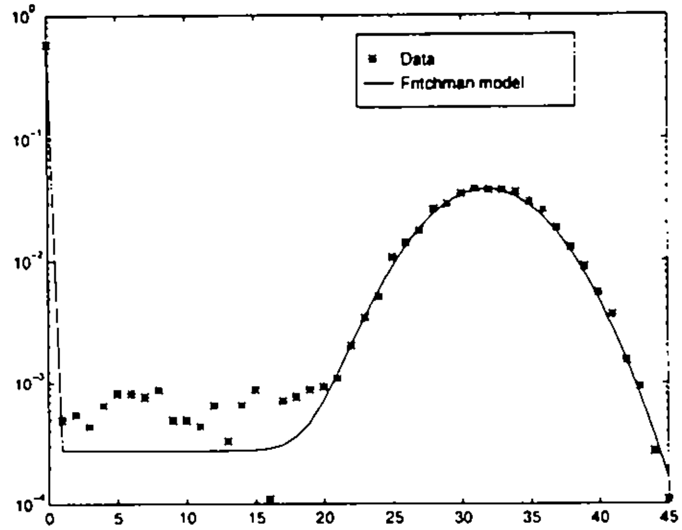

Figure 6.26: Fully Connected Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 070912.

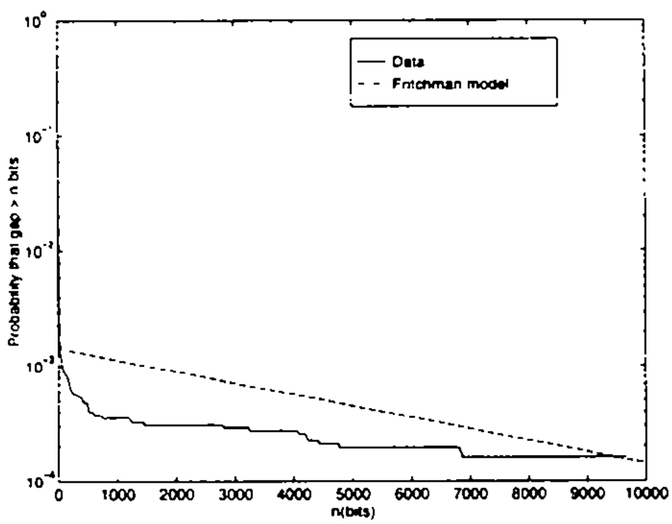

Figure 6.27: Fully Connected Fritchman model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 070912.

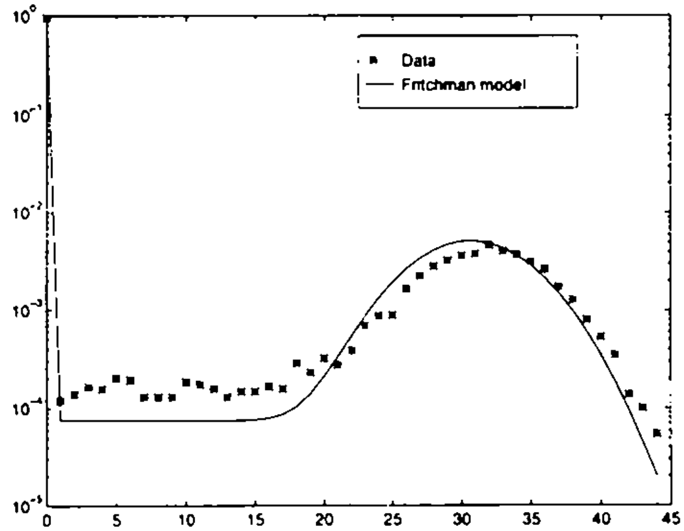

Figure 6.28: Fully Connected Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 071016.

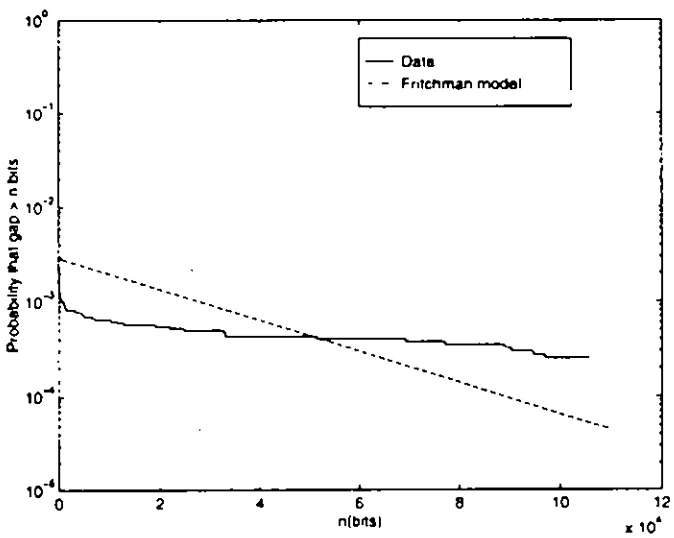

Figure 6.29: Fully Connected Fritchman model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 071016.

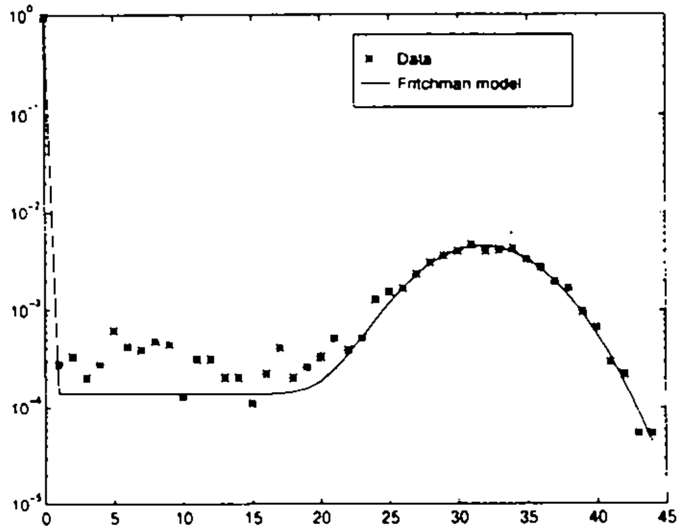

Figure 6.30: Fully Connected Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072406.

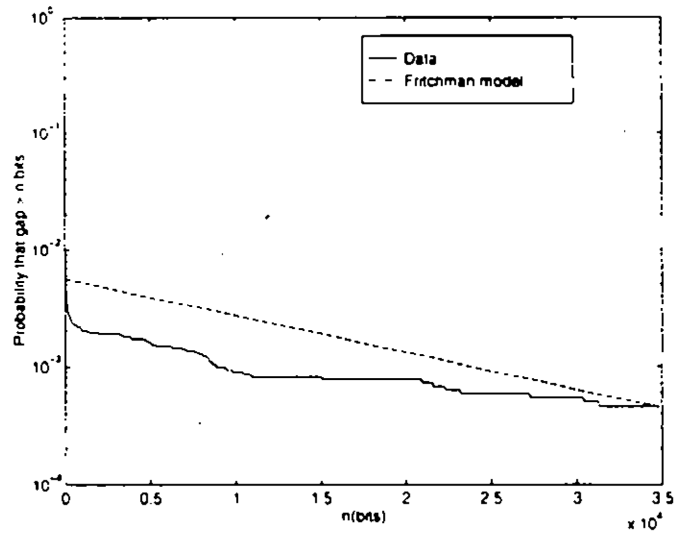

Figure 6.31: Fully Connected Fritchman model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 072406.

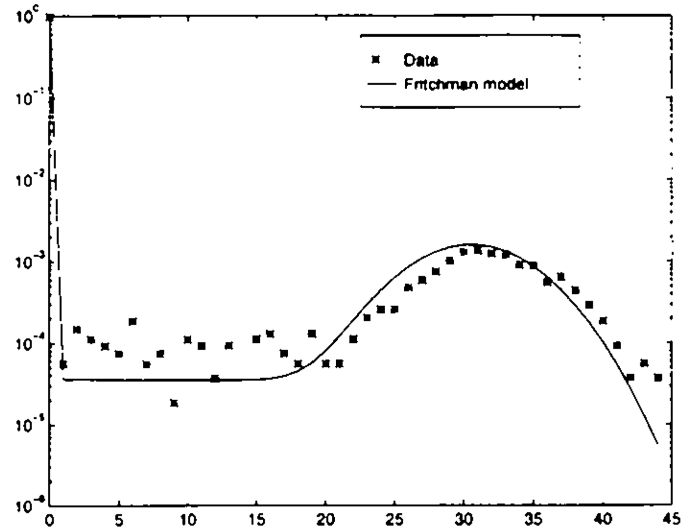

Figure 6.32: Fully Connected Fritchman Model for the Block Error Probabilities with Parameters Chosen to Match the Block Error Probabilities for Run 072411.

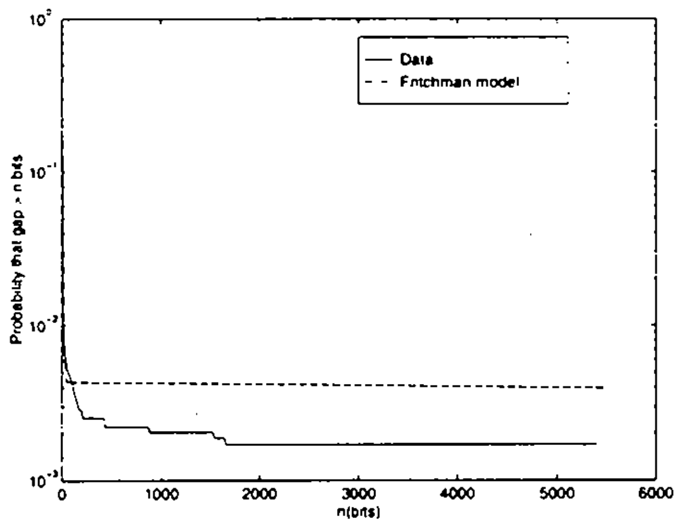

Figure 6.33: Fully Connected Fritchman model for the Error Gap Distribution with Parameters Chosen to Match the Block Error Probabilities for Run 072411.

Chapter 7

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

7.1 Conclusions

This thesis examined the applicability of four standard finite-state Markov chain models in characterizing the statistics of a synthetic error sequence typical of the Land Mobile Satellite Channel based on received signal level data collected using the ACTS Mobile Terminal. The models focused on in this thesis all had less than four states. While models with a larger number of states may more accurately represent the error process, the increased complexity makes these models unwieldy and limits the usefulness of such models.

The error sequence generated for this analysis was obtained using the information from fade and non-fade events. Applicability of the models was based on finding parameters used in fitting curves to the error gap distribution $u(n)$ and the average block error probabilities $\overline{P}(m,n)$.

The Gilbert model presented the simplest approach. It was a two state Markov chain representing BSC's where one state had a probability of bit error of zero, and another with a probability of bit error of h. Using this model, the larger gap distributions were approximated by $u(n)$. Simultaneously, the derived block error probabilities poorly approximated the $\overline{P}(m,n)$. Finding parameters that matched the block error probabilities resulted in good fits to $\overline{P}(m,n)$ and $u(n)$ that only fairly matched the gap distribution values for smaller n.

The Elliott model was an extension of the Gilbert model which made the two states in the chain BSC's with error probabilities of h and k. It took advantage of the methods used to obtain parameters in the Gilbert model. The improvement seen by using this model was that the block error probabilities were slightly better modeled than those obtained using the Gilbert model when given the error gap distribution.

The McCullough model was a two state Markov chain where transitions between states were only allowed after error bits, thus adding more structure to the Markov chain. The errors were characterized based on a threshold that determined whether or not the errors occuring were random or bursty. Modeling the block error probabilities given the error gap distribution showed poorer performance over the Gilbert and Elliott models, as did modeling the error gap distribution given the best fit to the block error probabilities. Choosing parameters to match the block error probabilities provided excellent fits at the expense of modeling the error gap distribution. Based on these parameters, a threshold was chosen for each run to determine when an error was due to a random error producing state or a bursty error producing state. Doing so led to fair fits to both $\overline{P}(m,n)$ and error gap distribution plots. The added structure inherent in this model is the main cause for the poor fits. The model assumes a non-zero probability of error in the random error producing state. However, in the synthetic error sequence, there are no random errors produced when the received pilot power levels are greater than the fading threshold. It is this mismatch between the data and the model that results in poor curve fits for both curves simultaneously using the McCullough model.

The Fritchman model was a three state Markov chain where the states represented individual bits instead of BSC's. Two error free states and one error state were chosen based on the error gap and error cluster distribution curves. The resulting models of the error gap distribution outperformed all other models. However only marginal success in matching the block error probabilities was obtained. This was due to restrictions on state transitions. With the restrictions removed, the block error probabilities were modeled at the expense of accuracy in modeling the error gap distribution.

In all of the models, either the error gap distribution or the block error probabilities was characterized accurately. Characterizing one of these two resulted in a degraded characterization of the other. None of the models were able to match both the error gap distribution and the block error probabilities with sufficient accuracy to be considered a complete model for both aspects of the channel, however the Elliott model seems to provide the best compromise.

7.2 Suggestions for Further Research

The models examined in this thesis were two state Markov chains that represented two different error producing processes, and a three state model where the states represented individual error bits. A Fritchman model with more than three states, particularly including extra error states, may provide more flexibility in modeling and may better represent the synthetic error sequence statistics. Also, the models examined in this thesis were based on a synthetic error sequence from the data taken using ACTS. Further investigation of these types of channels using a binary source would be of interest to further test these models. Also, the applicability to other sources of data, such as maritime channels, are also of interest.

Appendix A

COMPREHENSIVE RESULTS

A.1 Summary of Tables and Transition Probabilities

A.1.1 Tables for Gilbert, Elliott and McCullough Models

Table A.1: Gilbert Model Parameters Chosen to Match the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999983	0.676855	0.002977	0.678	1.75×10^{-5}	9.62×10^{-4}
070903	III	0.999930	0.588964	0.000498	0.589	6.97×10^{-5}	2.05×10^{-4}
070905	II	0.999906	0.622168	0.000450	0.622	9.31×10^{-5}	1.70×10^{-4}
070906	III	0.999965	0.595634	0.000684	0.596	3.47×10^{-5}	2.77×10^{-4}
070907	II	0.999947	0.621582	0.001244	0.622	5.26×10^{-5}	4.71×10^{-4}
070912	III	0.999901	0.5617968	0.000367	0.562	9.87×10^{-5}	1.61×10^{-4}
070914	III	0.999735	0.611230	0.000521	0.611	2.65×10^{-4}	2.02×10^{-4}
071016	II	0.999991	0.674779	0.000668	0.675	8.10×10^{-6}	2.18×10^{-4}
071017	II	0.999993	0.750000	0.000539	0.750	6.52×10^{-6}	1.35×10^{-4}
072405	II	0.999914	0.984838	0.001494	0.985	8.62×10^{-5}	2.24×10^{-5}
072406	II	0.999957	0.628417	0.001708	0.629	4.23×10^{-5}	6.35×10^{-4}
072407	II	0.999925	0.567187	0.000466	0.567	7.44×10^{-5}	2.02×10^{-4}
072408	II	0.999975	0.628417	0.001135	0.629	2.41×10^{-5}	4.22×10^{-4}
072409	II	0.999971	0.628710	0.003114	0.629	2.87×10^{-5}	1.16×10^{-3}
072410	II	0.999856	0.603027	0.000505	0.603	1.43×10^{-4}	2.00×10^{-4}
072411	II	0.999952	0.621386	0.002044	0.622	4.73×10^{-5}	7.74×10^{-4}
072412	II	0.999930	0.597949	0.000715	0.598	7.00×10^{-5}	2.88×10^{-4}

Table A.2: Gilbert Model Parameters Chosen to Match the Error Gap Distribution for $\eta = -10dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999643	0.729688	0.007109	0.731	3.59×10^{-4}	1.92×10^{-3}
070903	III	0.999933	0.639551	0.000577	0.640	6.65×10^{-5}	2.08×10^{-4}
070905	II	0.999912	0.679882	0.000534	0.680	8.71×10^{-5}	1.71×10^{-4}
070906	III	0.999964	0.665820	0.000859	0.666	3.55×10^{-5}	2.87×10^{-4}
070907	II	0.999961	0.696875	0.001279	0.697	3.82×10^{-5}	3.88×10^{-4}
070912	III	0.999905	0.635742	0.000416	0.636	9.50×10^{-5}	1.52×10^{-4}
070914	III	0.999988	0.637109	0.000362	0.637	1.16×10^{-4}	1.32×10^{-4}
071016	II	0.999992	0.730468	0.000660	0.731	7.35×10^{-6}	1.78×10^{-4}
071017	II	0.999991	0.835156	0.000721	0.835	8.16×10^{-6}	1.19×10^{-4}
072405	II	0.999930	0.695312	0.001679	0.696	6.96×10^{-5}	5.12×10^{-4}
072406	II	0.999971	0.740625	0.001492	0.741	2.88×10^{-5}	3.87×10^{-4}
072407	II	0.999931	0.595410	0.000467	0.596	6.82×10^{-5}	1.89×10^{-4}
072408	II	0.999976	0.747851	0.001561	0.748	2.35×10^{-5}	3.94×10^{-4}
072409	II	0.999979	0.834179	0.003275	0.835	2.03×10^{-5}	4.43×10^{-4}
072410	II	0.999861	0.655468	0.000588	0.656	1.40×10^{-4}	2.02×10^{-4}
072411	II	0.998816	0.613671	0.004925	0.615	1.19×10^{-3}	1.89×10^{-3}
072412	II	0.999940	0.663085	0.000719	0.663	6.00×10^{-5}	2.42×10^{-4}

Table A.3: Gilbert Model Parameters Chosen to Match the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999968	0.522116	3.50×10^{-3}	0.523	3.17×10^{-5}	1.67×10^{-3}
070903	III	0.999750	0.4987624	9.78×10^{-4}	0.499	2.50×10^{-4}	4.90×10^{-4}
070905	II	0.999580	0.493197	1.20×10^{-3}	0.493	4.2×10^{-4}	6.06×10^{-4}
070906	III	0.999904	0.4975809	1.62×10^{-3}	.497	9.55×10^{-5}	8.13×10^{-4}
070907	II	0.999892	0.491313	2.34×10^{-3}	0.490	1.08×10^{-4}	1.19×10^{-3}
070912	III	0.999773	0.501307	7.01×10^{-4}	0.501	2.27×10^{-4}	3.49×10^{-4}
070914	III	0.999428	0.501717	1.03×10^{-3}	0.502	5.72×10^{-4}	5.10×10^{-4}
071016	II	0.999962	0.520923	1.39×10^{-3}	0.520	3.80×10^{-5}	6.66×10^{-4}
071017	II	0.999899	0.241149	2.11×10^{-3}	0.521	1.01×10^{-4}	1.60×10^{-3}
072405	II	0.999693	0..496921	3.94×10^{-3}	0.497	3.08×10^{-4}	1.98×10^{-3}
072406	II	0.999926	0.501599	2.81×10^{-3}	0.502	7.32×10^{-5}	1.40×10^{-3}
072407	II	0.999845	0.4929315	7.62×10^{-4}	0.493	1.55×10^{-4}	3.86×10^{-4}
072408	II	0.999944	0.502804	1.97×10^{-3}	0.503	5.60×10^{-5}	9.79×10^{-4}
072409	II	0.999972	0.496521	4.49×10^{-3}	0.497	2.81×10^{-5}	2.26×10^{-3}
072410	II	0.999554	0.495626	9.99×10^{-4}	0.495	4.46×10^{-4}	5.03×10^{-4}
072411	II	0.999983	0.522043	2.09×10^{-3}	0.522	1.74×10^{-5}	9.96×10^{-4}
072412	II	0.999856	0.502513	1.34×10^{-3}	0.502	1.43×10^{-4}	6.65×10^{-4}

Table A.4: Gilbert Model Parameters Chosen to Match the Block Error Probabilities for $\eta = -10dB$.

RUN	CAT.	J	L	A	h	P_{12}	P_{21}
070901	II	0.999962	0.522299	7.85×10^{-3}	0.524	3.81×10^{-5}	3.75×10^{-3}
070903	III	0.999696	0.502852	1.47×10^{-3}	0.503	3.04×10^{-4}	7.29×10^{-4}
070905	II	0.999542	0.496278	1.82×10^{-3}	0.496	4.58×10^{-4}	9.14×10^{-4}
070906	III	0.999886	0.487534	2.43×10^{-3}	.488	1.14×10^{-4}	1.24×10^{-3}
070907	II	0.999886	0.498085	3.41×10^{-3}	0.498	1.14×10^{-4}	1.71×10^{-3}
070912	III	0.999682	0.499389	1.06×10^{-3}	0.499	3.17×10^{-4}	5.33×10^{-4}
070914	III	0.999556	0.501861	9.83×10^{-4}	0.502	4.44×10^{-4}	4.89×10^{-4}
071016	II	0.999533	0.498300	2.40×10^{-3}	0.498	4.68×10^{-4}	1.20×10^{-3}
071017	II	0.999946	0.522300	2.81×10^{-3}	0.523	5.37×10^{-5}	1.34×10^{-3}
072405	II	0.999785	0.496824	3.68×10^{-3}	0.497	2.15×10^{-4}	1.85×10^{-3}
072406	II	0.999918	0.490546	3.96×10^{-3}	0.491	8.14×10^{-5}	2.01×10^{-3}
072407	II	0.999836	0.501693	8.70×10^{-4}	0.434	1.64×10^{-4}	4.34×10^{-4}
072408	II	0.999919	0.499296	4.19×10^{-3}	0.500	8.10×10^{-5}	2.10×10^{-3}
072409	II	0.999961	0.493516	9.43×10^{-3}	0.495	3.83×10^{-5}	4.77×10^{-3}
072410	II	0.999433	0.497192	1.58×10^{-3}	0.497	5.67×10^{-4}	7.95×10^{-4}
072411	II	0.999981	0.522071	3.04×10^{-3}	0.522	1.86×10^{-5}	1.45×10^{-3}
072412	II	0.998277	0.497166	1.84×10^{-3}	0.497	1.72×10^{-4}	9.24×10^{-4}

Table A.5: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	J	L	A
070901	II	0.999983	0.676855	0.002977
070903	III	0.999930	0.58896	0.000498
070905	II	0.999906	0.622168	0.000450
070906	III	0.999965	0.595634	0.000684
070907	II	0.999947	0.621580	0.001244
070912	III	0.999901	0.561796	0.000367
070914	III	0.999735	0.611230	0.000521
071016	II	0.999991	0.674779	0.000668
071017	II	0.999993	0.750000	0.000539
072405	II	0.999914	0.984838	0.001494
072406	II	0.999957	0.628417	0.001708
072407	II	0.999925	0.567187	0.000466
072408	II	0.999975	0.628417	0.001135
072409	II	0.999971	0.628710	0.003114
072410	II	0.999856	0.603027	0.000505
072411	II	0.999952	0.621386	0.002044
072412	II	0.999930	0.597949	0.000715

Table A.6: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Error Gap Distribution for $\eta = -10dB$.

RUN	CAT.	J	L	A
070901	II	0.999643	0.729688	0.007109
070903	III	0.999933	0.639551	0.000577
070905	II	0.999912	0.679882	0.000534
070906	III	0.999964	0.665820	0.000859
070907	II	0.999961	0.696875	0.001279
070912	III	0.999905	0.635742	0.000416
070914	III	0.999988	0.637109	0.000362
071016	II	0.999992	0.730468	0.000660
071017	II	0.999991	0.835156	0.000721
072405	II	0.999930	0.695312	0.001679
072406	II	0.999971	0.740625	0.001492
072407	II	0.999931	0.595410	0.000467
072408	II	0.999976	0.747851	0.001561
072409	II	0.999979	0.834179	0.003275
072410	II	0.999861	0.655468	0.000588
072411	II	0.998816	0.613671	0.004925
072412	II	0.999940	0.663085	0.000719

Table A.7: Elliott Model Parameters for the Block Error Probabilities Based on Best Fit to the Error Gap Distribution for $\eta = -6dB$.

RUN	CAT.	h	P_{12}	P_{21}	k
070901	II	0.678	1.75×10^{-5}	9.62×10^{-4}	0.999999
070903	III	0.589	6.97×10^{-5}	2.05×10^{-4}	0.999996
070905	II	0.622	9.31×10^{-5}	1.70×10^{-4}	0.999984
070906	III	0.596	3.47×10^{-5}	2.77×10^{-4}	0.999998
070907	II	0.622	5.26×10^{-5}	4.71×10^{-4}	0.999997
070912	III	0.562	9.87×10^{-5}	1.61×10^{-4}	0.999994
070914	III	0.611	2.65×10^{-4}	2.02×10^{-4}	0.999999
071016	II	0.675	8.10×10^{-6}	2.18×10^{-4}	0.999999
071017	II	0.750	6.52×10^{-6}	1.35×10^{-4}	0.999997
072405	II	0.985	8.62×10^{-5}	2.24×10^{-5}	0.517587
072406	II	0.629	4.23×10^{-5}	6.35×10^{-4}	0.999999
072407	II	0.567	7.44×10^{-5}	2.02×10^{-4}	0.999994
072408	II	0.629	2.41×10^{-5}	4.22×10^{-4}	0.999994
072409	II	0.629	2.87×10^{-5}	1.16×10^{-3}	0.999999
072410	II	0.603	1.43×10^{-4}	2.00×10^{-4}	0.999999
072411	II	0.622	4.73×10^{-5}	7.74×10^{-4}	0.999999
072412	II	0.598	7.00×10^{-5}	2.88×10^{-4}	0.999996

Table A.8: Elliott Model Parameters for the Block Error Probabilities Based on Best Fit to the Error Gap Distribution for $\eta = -10dB$.

RUN	CAT.	h	P_{12}	P_{21}	k
070901	II	.731	3.59×10^{-4}	1.92×10^{-3}	0.999999
070903	III	0.640	6.65×10^{-5}	2.08×10^{-4}	0.999991
070905	II	0.680	8.71×10^{-5}	1.71×10^{-4}	0.999988
070906	III	0.666	3.55×10^{-5}	2.87×10^{-4}	0.999995
070907	II	0.697	3.82×10^{-5}	3.88×10^{-4}	0.999998
070912	III	0.636	9.50×10^{-5}	1.52×10^{-4}	0.999988
070914	III	0.637	1.16×10^{-4}	1.32×10^{-4}	0.999985
071016	II	0.731	7.35×10^{-6}	1.78×10^{-4}	0.999998
071017	II	0.835	8.16×10^{-6}	1.19×10^{-4}	0.999999
072405	II	0.696	6.96×10^{-5}	5.12×10^{-4}	0.999997
072406	II	0.741	2.88×10^{-5}	3.87×10^{-4}	0.999999
072407	II	0.596	6.82×10^{-5}	1.89×10^{-4}	0.999994
072408	II	0.748	2.35×10^{-5}	3.94×10^{-4}	0.999998
072409	II	0.835	2.03×10^{-5}	4.43×10^{-4}	0.999999
072410	II	0.656	1.40×10^{-4}	2.02×10^{-4}	0.999988
072411	II	0.615	1.19×10^{-3}	1.89×10^{-3}	0.999999
072412	II	0.663	6.00×10^{-5}	2.42×10^{-4}	0.999995

Table A.9: Elliott Model Parameters for Block Error Probabilities Based on Best Fit to the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	P_{12}	P_{21}	h	k
070901	II	3.16×10^{-5}	1.84×10^{-3}	0.515	0.995625
070903	III	2.87×10^{-4}	7.35×10^{-4}	0.502	0.999998
070905	II	4.22×10^{-4}	5.51×10^{-4}	0.497	0.999999
070906	III	8.35×10^{-5}	4.25×10^{-4}	0.508	0.999999
070907	II	1.38×10^{-4}	1.18×10^{-3}	0.512	0.999999
070912	III	2.31×10^{-4}	3.74×10^{-4}	0.494	0.999999
070914	III	3.84×10^{-3}	4.01×10^{-4}	0.498	0.999999
071016	II	3.25×10^{-3}	5.40×10^{-5}	0.520	0.999999
071017	II	1.15×10^{-4}	4.40×10^{-4}	0.520	0.999999
072405	II	2.94×10^{-4}	1.33×10^{-3}	0.518	0.999999
072406	II	1.19×10^{-4}	2.35×10^{-3}	0.523	0.999999
072407	II	1.53×10^{-4}	3.96×10^{-4}	0.491	0.999998
072408	II	2.90×10^{-5}	2.72×10^{-4}	0.521	0.999998
072409	II	6.00×10^{-6}	1.31×10^{-4}	0.496	0.999977
072410	II	4.48×10^{-4}	4.85×10^{-4}	0.497	0.999999
072411	II	2.16×10^{-3}	2.70×10^{-5}	0.517	0.999999
072412	II	1.39×10^{-4}	6.01×10^{-4}	0.508	0.999999

Table A.10: Elliott Model Parameters for the Block Error Probabilities Based on Best Fit to the Block Error Probabilities for $\eta = -10dB$.

RUN	CAT.	h	P_{12}	P_{21}	k
070901	II	0.525	3.74×10^{-5}	3.62×10^{-3}	0.999997
070903	III	0.507	2.90×10^{-4}	3.62×10^{-3}	0.999999
070905	II	0.501	4.59×10^{-4}	8.21×10^{-4}	0.999999
070906	III	0.506	1.34×10^{-4}	1.42×10^{-3}	0.999999
070907	II	0.498	2.69×10^{-3}	1.93×10^{-4}	0.999999
070912	III	0.496	2.95×10^{-3}	3.46×10^{-4}	0.999999
070914	III	0.522	4.12×10^{-5}	7.81×10^{-4}	0.999999
071016	II	0.526	5.29×10^{-5}	1.27×10^{-3}	0.999999
071017	II	0.526	4.29×10^{-5}	1.27×10^{-3}	0.999999
072405	II	0.526	2.21×10^{-4}	1.22×10^{-3}	0.999999
072406	II	0.518	9.56×10^{-5}	1.84×10^{-3}	0.999999
072407	II	0.501	2.35×10^{-4}	1.07×10^{-3}	0.999999
072408	II	0.519	1.13×10^{-4}	3.22×10^{-3}	0.999999
072409	II	0.495	3.83×10^{-5}	4.78×10^{-3}	0.999997
072410	II	0.497	6.14×10^{-4}	9.77×10^{-4}	0.999999
072411	II	0.505	1.87×10^{-5}	1.72×10^{-3}	0.999999
072412	II	0.511	1.65×10^{-4}	7.26×10^{-4}	0.999999

Table A.11: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	J	L	A
070901	II	0.999968	0.514388	0.003790
070903	III	0.999713	0.501330	0.001480
070905	II	0.999578	0.496871	0.001100
070906	III	0.999916	0.508292	0.000865
070907	II	0.999862	0.512238	0.002430
070912	III	0.999769	0.494674	0.000741
070914	III	0.996158	0.497946	0.000808
071016	II	0.996747	0.520330	0.000114
071017	II	0.999885	0.520071	0.000917
072405	II	0.999706	0.517864	0.002760
072406	II	0.999881	0.522578	0.004910
072407	II	0.999847	0.491140	0.000779
072408	II	0.999971	0.520914	0.000568
072409	II	0.999994	0.496828	0.000260
072410	II	0.999552	0.497099	0.000966
072411	II	0.997836	0.517698	0.000056
072412	II	0.999861	0.508595	0.001220

Table A.12: Elliott Model Parameters for the Error Gap Distribution Based on Best Fit to the Block Error Probabilities for $\eta = -10dB$.

RUN	CAT.	J	L	A
070901	II	0.999962	0.523985	0.007600
070903	III	0.997099	0.506852	0.002240
070905	II	0.999541	0.501091	0.001650
070906	III	0.999865	0.505938	0.002870
070907	II	0.999898	0.520424	0.002400
070912	III	0.997306	0.498621	0.000589
070914	III	0.997050	0.496803	0.000694
071016	II	0.999958	0.522512	0.001640
071017	II	0.999947	0.525387	0.002670
072405	II	0.999778	0.524840	0.002570
072406	II	0.999904	0.517693	0.003810
072407	II	0.999765	0.500806	0.002140
072408	II	0.999887	0.518126	0.006680
072409	II	0.999961	0.493541	0.009440
072410	II	0.999386	0.497184	0.001950
072411	II	0.999981	0.504725	0.003480
072412	II	0.999834	0.510540	0.001480

Table A.13: McCullough Model Parameters Based On $k = 10, \eta = -6dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.991786	0.013699	0.991740	0.008260	0.510873	0.000065
070903	III	0.997688	0.005435	0.997680	0.002320	0.504467	0.000592
070905	II	0.997231	0.008115	0.997216	0.002780	0.502659	0.000930
070906	III	0.996598	0.002817	0.996600	0.003400	0.502499	0.000201
070907	II	0.995652	0.004250	0.995652	0.000435	0.505040	0.000258
070912	III	0.998002	0.004032	0.997998	0.002002	0.503224	0.000779
070914	III	0.996862	0.005556	0.996854	0.003150	0.503145	0.001743
071016	II	0.995449	0.005464	0.995445	0.004555	0.501682	0.000109
071017	II	0.994649	0.011628	0.994615	0.005390	0.501043	0.000119
072405	II	0.992029	0.003451	0.992065	0.007940	0.504465	0.000631
072406	II	0.993525	0	0.993567	0.006433	0.502845	0.000169
072407	II	0.997904	0.001727	0.997905	0.002100	0.503620	0.000412
072408	II	0.995127	0.013575	0.995084	0.004920	0.500528	0.000138
072409	II	0.992682	0.012195	0.992646	0.007350	0.501537	0.000048
072410	II	0.997265	0.005208	0.997258	0.002740	0.502578	0.001132
072411	II	0.994264	0.014286	0.994214	0.005786	0.502906	0.000047
072412	II	0.996851	0.005367	0.996844	0.003160	0.503137	0.000349

Table A.14: McCullough Model Parameters Based On $k = 10, \eta = -10dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.983091	0.024691	0.982957	0.017043	0.507423	0.000079
070903	III	0.996688	0.004323	0.996685	0.003315	0.503118	0.000071
070905	II	0.996357	0.006568	0.996347	0.003653	0.503124	0.000930
070906	III	0.995502	0.002639	0.995510	0.004490	0.502774	0.000211
070907	II	0.993915	0.00482	0.993926	0.006074	0.500884	0.000261
070912	III	0.997126	0.004320	0.997125	0.002875	0.502795	0.000911
070914	III	0.997126	0.003175	0.997125	0.002875	0.502795	0.000106
071016	II	0.994667	0.002793	0.994681	0.005319	0.506233	0.000106
071017	II	0.992055	0.006981	0.992063	0.007937	0.504388	0.000131
072405	II	0.992230	0.004713	0.992253	0.007747	0.504209	0.000444
072406	II	0.991431	0.006969	0.991444	0.008556	0.505792	0.000170
072407	II	0.997381	0.004405	0.997376	0.002624	0.502991	0.000474
072408	II	0.990771	0.010101	0.990763	0.009247	0.505477	0.000182
072409	II	0.984820	0.018349	0.984772	0.015228	0.504515	0.000136
072410	II	0.996286	0.004286	0.996284	0.003716	0.503363	0.001266
072411	II	0.990501	0.023256	0.990367	0.009633	0.505700	0.000058
072412	II	0.995496	0.004491	0.995509	0.004491	0.504191	0.000415

Table A.15: McCullough Model Parameters Chosen to Match Error Gap Distribution for $\eta = -6$ dB.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.999876	0.958639	0.997023	0.002977	0.323145	0.000017
070903	III	0.999501	0.000283	0.999502	0.000498	0.411036	0.000069
070905	II	0.999549	0.000018	0.999549	0.000045	0.377832	0.000093
070906	III	0.999704	0.568000	0.999315	0.000684	0.404366	0.000034
070907	II	0.999682	0.777641	0.998756	0.001244	0.378418	0.000052
070912	III	0.999632	0.000035	0.999632	0.000368	0.438203	0.000098
070914	III	0.999478	0.000054	0.999478	0.000521	0.388769	0.000265
071016	II	0.999877	0.817678	0.999331	0.000669	0.325221	0.000008
071017	II	0.999891	0.799415	0.999461	0.000539	0.250000	0.000006
072405	II	0.997341	0.828900	0.999948	0.000051	0.001900	0.000005
072406	II	0.999722	0.837577	0.998291	0.001709	0.371583	0.000042
072407	II	0.999603	0.150029	0.999533	0.000466	0.432813	0.000074
072408	II	0.999812	0.834972	0.998865	0.001135	0.371583	0.000024
072409	II	0.999908	0.970712	0.996886	0.003114	0.371290	0.000028
072410	II	0.999494	0.000025	0.999494	0.000505	0.396973	0.000143
072411	II	0.999931	0.966222	0.997956	0.002044	0.378614	0.000047
072412	II	0.999579	0.412376	0.999284	0.000715	0.402051	0.000069

Table A.16: McCullough Model Parameters Chosen to Match Error Gap Distribution for $\eta = -10$ dB.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.999920	0.988925	0.992891	0.007109	0.270312	0.000357
070903	III	0.999422	0.000830	0.999422	0.000577	0.000066	0.360449
070905	II	0.999469	0.000011	0.999465	0.000534	0.320118	0.000087
070906	III	0.999689	0.638691	0.999140	0.000860	0.334180	0.000035
070907	II	0.999716	0.778722	0.998720	0.001280	0.303125	0.000038
070912	III	0.999583	0.000052	0.999583	0.000416	0.364258	0.000095
070914	III	0.999637	0.000031	0.999637	0.000363	0.362891	0.000116
071016	II	0.999874	0.810827	0.999339	0.000660	0.269532	0.000007
071017	II	0.999999	0.999999	0.999278	0.000721	0.164844	0.000008
072405	II	0.999512	0.710026	0.998320	0.001679	0.304687	0.000069
072406	II	0.999832	0.887818	0.998507	0.001492	0.259375	0.000028
072407	II	0.999573	0.087451	0.999532	0.000046	0.404589	0.000068
072408	II	0.999817	0.883052	0.998438	0.001562	0.252148	0.000023
072409	II	0.999999	0.999999	0.996724	0.003275	0.165821	0.000020
072410	II	0.999411	0.000028	0.999412	0.000588	0.344532	0.000140
072411	II	0.999932	0.986290	0.995074	0.004926	0.386328	0.001180
072412	II	0.999567	0.398398	0.999281	0.000719	0.336914	0.000060

Table A.17: McCullough Model Parameters Chosen to Match Block Error Probabilities for $\eta = -6dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.990553	0.233954	0.016470	0.983530	0.503795	3.22×10^{-10}
070903	III	.998084	0.175897	0.348031	0.651969	0.501443	2.72×10^{-10}
070905	II	0.997680	0.034927	0.425552	0.574448	0.506998	2.89×10^{-10}
070906	III	0.996833	0.225946	0.110316	0.889684	0.502786	2.56×10^{-10}
070907	II	0.995466	0.129568	0.089953	0.910047	0.509191	2.46×10^{-12}
070912	III	0.9986169	0.159234	0.402314	0.597686	0.498854	2.54×10^{-10}
070914	III	0.998013	0.109217	0.545000	0.455000	0.498464	2.54×10^{-10}
071016	II	0.9965864	0.1072376	0.048623	0.951376	0.498145	3.00×10^{-10}
071017	II	0.995744	0.327693	0.045394	0.954606	0.497824	3.64×10^{-10}
072405	II	0.992538	0.231381	0.151061	0.848939	0.503814	2.29×10^{-10}
072406	II	0.999636	6.93×10^{-11}	0.769358	0.230642	0.497160	0.000155
072407	II	0.998926	0.165302	0.401982	0.598018	0.503452	2.54×10^{-10}
072408	II	0.998256	0.112784	0.121366	0.878634	0.489798	3.30×10^{-16}
072409	II	0.991515	0.054639	0.014099	0.985901	0.503670	3.71×10^{-10}
072410	II	0.998061	0.158672	0.485162	0.514838	0.504510	1.97×10^{-10}
072411	II	0.994407	0.296454	0.014720	0.98528	0.507189	3.00×10^{-10}
072412	II	0.998737	0.306878	0.374730	0.625270	0.488872	4.61×10^{-12}

Table A.18: McCullough Model Parameters Chosen to Match Block Error Probabilities for $\eta = -10dB$.

RUN	CAT.	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	0.978859	0.000005	0.513091	0.486909	0.513091	5.49×10^{-10}
070903	III	0.997148	0.174233	0.307966	0.692034	0.497394	2.75×10^{-10}
070905	II	0.996501	0.175192	0.353318	0.646682	0.504075	2.65×10^{-10}
070906	III	0.995302	0.131262	0.091013	0.908986	0.512912	4.61×10^{-10}
070907	II	0.993496	0.177991	0.069665	0.930335	0.502477	3.80×10^{-10}
070912	III	0.997899	0.151717	0.383078	0.616922	0.501282	2.41×10^{-10}
070914	III	0.998085	0.155954	0.490678	0.509322	0.498301	2.46×10^{-10}
071016	II	0.995331	0.319681	0.040178	0.959821	0.502211	4.79×10^{-10}
071017	II	0.992791	0.217516	0.033915	0.966085	0.498737	1.76×10^{-10}
072405	II	0.992999	0.194015	0.116011	0.883989	0.503879	3.29×10^{-10}
072406	II	0.992456	0.278624	0.046694	0.953306	0.510138	2.34×10^{-10}
072407	II	0.998231	0.144657	0.275670	0.724330	0.501101	2.91×10^{-10}
072408	II	0.992005	0.205480	0.041981	0.958019	0.501412	3.52×10^{-10}
072409	II	0.982937	0.404707	0.010328	0.989671	0.507684	6.37×10^{-11}
072410	II	0.996954	0.160877	0.437181	0.562819	0.503010	2.47×10^{-10}
072411	II	0.992555	0.062791	0.011904	0.988096	0.497351	2.51×10^{-11}
072412	II	0.996416	0.191363	0.166239	0.833761	0.503241	3.55×10^{-10}

Table A.19: McCullough Model Parameters Using Best $k, \eta = -6dB$.

RUN	CAT.	k	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	740	0.998736	0.076923	0.998623	0.001380	0.355509	0.000011
070903	III	10	0.997688	0.005435	0.997680	0.002320	0.504467	0.000592
070905	II	20	0.998709	0.013436	0.998693	0.001307	0.498883	0.000439
070906	III	430	0.999414	0.015873	0.999405	0.000595	0.465702	0.000036
070907	II	460	0.999257	0.012658	0.999248	0.000752	0.447668	0.000045
070912	III	20	0.999229	0.010309	0.999222	0.000778	0.499994	0.000305
070914	III	20	0.998785	0.000000	0.998786	0.001214	0.498143	0.000682
071016	II	1450	0.999613	0.030303	0.999601	0.000399	0.434376	0.000010
071017	II	730	0.999529	0.021277	0.999519	0.000481	0.431400	0.000011
072405	II	210	0.998469	0.007092	0.998460	0.001540	0.451716	0.000125
072406	II	740	0.998938	0.000000	0.998939	0.001061	0.437516	0.000028
072407	II	40	0.999336	0.000000	0.999337	0.000663	0.499340	0.000131
072408	II	2000	0.999284	0.000000	0.999285	0.000715	0.418303	0.000020
072409	II	560	0.998636	0.058824	0.998547	0.001453	0.399390	0.000009
072410	II	20	0.998863	0.008299	0.998854	0.001146	0.498438	0.000477
072411	II	1660	0.999615	0.000000	0.999615	0.000385	0.452093	0.000044
072412	II	300	0.999440	0.000000	0.999440	0.000560	0.478072	0.000063

Table A.20: McCullough Model Parameters Using Best $k, \eta = -6dB$.

RUN	CAT.	k	q_{11}	q_{22}	Q_1	Q_2	P_1	P_2
070901	II	350	0.997222	0.066667	0.997017	0.0029983	0.328675	0.000014
070903	III	90	0.999066	0.015075	0.999053	0.000947	0.491363	0.000205
070905	II	70	0.998704	0.004608	0.998700	0.001300	0.492767	0.000335
070906	III	280	0.999215	0.014706	0.999204	0.000796	0.461322	0.000038
070907	II	420	0.999026	0.013158	0.999014	0.000986	0.419724	0.000043
070912	III	40	0.999001	0.009009	0.998993	0.001007	0.496958	0.000322
070914	III	20	0.998807	0.005525	0.998802	0.001197	0.497459	0.000519
071016	II	20	0.997326	0.005525	0.997318	0.002682	0.499105	0.000053
071017	II	660	0.999347	0.020408	0.999344	0.000666	0.410704	0.000011
072405	II	270	0.998536	0.004950	0.998531	0.001469	0.450783	0.000084
072406	II	350	0.998802	0.023810	0.998773	0.001227	0.428962	0.000025
072407	II	80	0.999355	0.005917	0.999352	0.000648	0.495418	0.000118
072408	II	550	0.998969	0.028571	0.998939	0.001061	0.424489	0.000021
072409	II	540	0.998305	0.071429	0.998167	0.001833	0.314259	0.000017
072410	II	30	0.998508	0.001776	0.998507	0.001493	0.496718	0.000514
072411	II	860	0.999206	0.111111	0.999093	0.000907	0.380066	0.000005
072412	II	320	0.99933	0.000000	0.999330	0.000670	0.466830	0.000063

A.1.2 Transition Probabilities for Fritchman Model

The transition probabilities for the simplified model that give the closest fits to the error gap distributions for $\eta = -6\text{dB}$ are

$$P_{070901} = \begin{bmatrix} 0.995276 & 0 & 0.004723 \\ 0 & 0.999982 & 0.000018 \\ 0.008601 & 0.002981 & 0.988417 \end{bmatrix} \tag{A.1}$$

$$P_{070903} = \begin{bmatrix} 0.997670 & 0 & 0.002330 \\ 0 & 0.999930 & 0.000070 \\ 0.001155 & 0.000499 & 0.998345 \end{bmatrix} \tag{A.2}$$

$$P_{070905} = \begin{bmatrix} 0.997955 & 0 & 0.002045 \\ 0 & 0.999906 & 0.000094 \\ 0.001277 & 0.000450 & 0.998273 \end{bmatrix} \tag{A.3}$$

$$P_{070906} = \begin{bmatrix} 0.999792 & 0 & 0.000208 \\ 0 & 0.999976 & 0.000024 \\ 0.000876 & 0.000447 & 0.998677 \end{bmatrix} \tag{A.4}$$

$$P_{070907} = \begin{bmatrix} 0.996351 & 0 & 0.003649 \\ 0 & 0.999947 & 0.000053 \\ 0.002391 & 0.001246 & 0.996363 \end{bmatrix} \tag{A.5}$$

$$P_{070912} = \begin{bmatrix} 0.993361 & 0 & 0.006639 \\ 0 & 0.999869 & 0.000131 \\ 0.000924 & 0.000446 & 0.998630 \end{bmatrix} \tag{A.6}$$

$$P_{070914} = \begin{bmatrix} 0.989982 & 0 & 0.010018 \\ 0 & 0.999734 & 0.000266 \\ 0.002141 & 0.000522 & 0.997337 \end{bmatrix} \tag{A.7}$$

$$P_{071016} = \begin{bmatrix} 0.964321 & 0 & 0.035679 \\ 0 & 0.999991 & 0.000009 \\ 0.000717 & 0.000685 & 0.998598 \end{bmatrix} \tag{A.8}$$

$$P_{071017} = \begin{bmatrix} 0.999686 & 0 & 0.000314 \\ 0 & 0.999993 & 0.000007 \\ 0.001156 & 0.000539 & 0.998305 \end{bmatrix} \tag{A.9}$$

$$P_{072405} = \begin{bmatrix} 0.996924 & 0 & 0.003076 \\ 0 & 0.999914 & 0.000086 \\ 0.005263 & 0.001490 & 0.993247 \end{bmatrix} \tag{A.10}$$

$$P_{072406} = \begin{bmatrix} 0.999999 & 0 & 0.000001 \\ 0 & 0.999922 & 0.000078 \\ 0.000350 & 0.001574 & 0.998076 \end{bmatrix} \tag{A.11}$$

$$P_{072407} = \begin{bmatrix} 0.998502 & 0 & 0.001498 \\ 0 & 0.999925 & 0.000075 \\ 0.001708 & 0.000465 & 0.997827 \end{bmatrix} \tag{A.12}$$

$$P_{072408} = \begin{bmatrix} 0.998860 & 0 & 0.001140 \\ 0 & 0.999975 & 0.000025 \\ 0.002717 & 0.001135 & 0.996148 \end{bmatrix} \tag{A.13}$$

$$P_{072409} = \begin{bmatrix} 0.988439 & 0 & 0.011561 \\ 0 & 0.999971 & 0.000029 \\ 0.012496 & 0.003124 & 0.984380 \end{bmatrix} \tag{A.14}$$

$$P_{072410} = \begin{bmatrix} 0.997270 & 0 & 0.002730 \\ 0 & 0.999858 & 0.000142 \\ 0.001049 & 0.000505 & 0.998446 \end{bmatrix} \tag{A.15}$$

$$P_{072411} = \begin{bmatrix} 0.621006 & 0 & 0.378994 \\ 0 & 0.999944 & 0.000056 \\ 0.008127 & 0.002113 & 0.989760 \end{bmatrix} \tag{A.16}$$

$$P_{072412} = \begin{bmatrix} 0.995449 & 0 & 0.004551 \\ 0 & 0.999929 & 0.000071 \\ 0.002503 & 0.000716 & 0.996781 \end{bmatrix}. \tag{A.17}$$

The transition probabilities for the simplified model that give the closest fits to the error gap distributions for $\eta = -10$dB are

$$P_{070901} = \begin{bmatrix} 0.988428 & 0 & 0.011572 \\ 0 & 0.999644 & 0.000356 \\ 0.025711 & 0.007090 & 0.967199 \end{bmatrix} \tag{A.18}$$

$$P_{070903} = \begin{bmatrix} 0.996122 & 0 & 0.003878 \\ 0 & 0.999933 & 0.000067 \\ 0.002026 & 0.000578 & 0.997396 \end{bmatrix} \tag{A.19}$$

$$P_{070905} = \begin{bmatrix} 0.997965 & 0 & 0.002035 \\ 0 & 0.999912 & 0.000088 \\ 0.001717 & 0.000535 & 0.997748 \end{bmatrix} \tag{A.20}$$

$$P_{070906} = \begin{bmatrix} 0.997036 & 0 & 0.002964 \\ 0 & 0.999964 & 0.000036 \\ 0.002862 & 0.000860 & 0.996278 \end{bmatrix} \tag{A.21}$$

$$P_{070907} = \begin{bmatrix} 0.996718 & 0 & 0.003282 \\ 0 & 0.999961 & 0.000039 \\ 0.003223 & 0.001280 & 0.995497 \end{bmatrix} \tag{A.22}$$

$$P_{070912} = \begin{bmatrix} 0.993851 & 0 & 0.006149 \\ 0 & 0.999904 & 0.000096 \\ 0.002529 & 0.000416 & 0.997055 \end{bmatrix} \tag{A.23}$$

$$P_{070914} = \begin{bmatrix} 0.993632 & 0 & 0.006368 \\ 0 & 0.999883 & 0.000117 \\ 0.001935 & 0.000363 & 0.997702 \end{bmatrix} \tag{A.24}$$

$$P_{071016} = \begin{bmatrix} 0.998866 & 0 & 0.001134 \\ 0 & 0.999992 & 0.000008 \\ 0.001538 & 0.000660 & 0.997802 \end{bmatrix} \tag{A.25}$$

$$P_{071017} = \begin{bmatrix} 0.998987 & 0 & 0.001013 \\ 0 & 0.999991 & 0.000009 \\ 0.001707 & 0.000721 & 0.997572 \end{bmatrix} \tag{A.26}$$

$$P_{072405} = \begin{bmatrix} 0.997271 & 0 & 0.002729 \\ 0 & 0.999930 & 0.000070 \\ 0.004528 & 0.001679 & 0.997393 \end{bmatrix} \tag{A.27}$$

$$P_{072406} = \begin{bmatrix} 0.997963 & 0 & 0.002037 \\ 0 & 0.999971 & 0.000029 \\ 0.004634 & 0.001493 & 0.993873 \end{bmatrix} \tag{A.28}$$

$$P_{072407} = \begin{bmatrix} 0.998355 & 0 & 0.001645 \\ 0 & 0.999931 & 0.000069 \\ 0.002053 & 0.000467 & 0.997480 \end{bmatrix} \tag{A.29}$$

$$P_{072408} = \begin{bmatrix} 0.995908 & 0 & 0.004092 \\ 0 & 0.999976 & 0.000024 \\ 0.005885 & 0.001563 & 0.992552 \end{bmatrix} \tag{A.30}$$

$$P_{072409} = \begin{bmatrix} 0.994734 & 0 & 0.005266 \\ 0 & 0.999979 & 0.000021 \\ 0.015073 & 0.003278 & 0.981649 \end{bmatrix} \tag{A.31}$$

$$P_{072410} = \begin{bmatrix} 0.994421 & 0 & 0.005579 \\ 0 & 0.999860 & 0.000140 \\ 0.003055 & 0.000588 & 0.996357 \end{bmatrix} \tag{A.32}$$

$$P_{072411} = \begin{bmatrix} 0.904443 & 0 & 0.095557 \\ 0 & 0.998813 & 0.001187 \\ 0.067598 & 0.004933 & 0.927469 \end{bmatrix} \tag{A.33}$$

$$P_{072412} = \begin{bmatrix} 0.996457 & 0 & 0.003543 \\ 0 & 0.999940 & 0.000060 \\ 0.002641 & 0.000719 & 0.996640 \end{bmatrix}. \tag{A.34}$$

The transition probabilities for the simplified model that give the best fits to $\overline{P}(m,n)$ for $\eta = -6$ dB are

$$P_{070901} = \begin{bmatrix} 0.966955 & 0 & 0.033045 \\ 0 & 0.999948 & 0.000052 \\ 0.009186 & 0.002988 & 0.987825 \end{bmatrix} \tag{A.35}$$

$$P_{070903} = \begin{bmatrix} 0.970926 & 0 & 0.029074 \\ 0 & 0.999236 & 0.000764 \\ 0.003382 & 0.001153 & 0.995464 \end{bmatrix} \tag{A.36}$$

$$P_{070905} = \begin{bmatrix} 0.986257 & 0 & 0.013743 \\ 0 & 0.998400 & 0.001600 \\ 0.001637 & 0.001059 & 0.997302 \end{bmatrix} \tag{A.37}$$

$$P_{070906} = \begin{bmatrix} 0.977238 & 0 & 0.022762 \\ 0 & 0.999504 & 0.000496 \\ 0.000760 & 0.000297 & 0.998943 \end{bmatrix} \tag{A.38}$$

$$P_{070907} = \begin{bmatrix} 0.999723 & 0 & 0.000277 \\ 0 & 0.994230 & 0.005770 \\ 0.002503 & 0.001249 & 0.996247 \end{bmatrix} \tag{A.39}$$

$$P_{070912} = \begin{bmatrix} 0.967918 & 0 & 0.032082 \\ 0 & 0.999943 & 0.000057 \\ 0.003725 & 0.000013 & 0.996262 \end{bmatrix} \quad (A.40)$$

$$P_{070914} = \begin{bmatrix} 0.970825 & 0 & 0.029175 \\ 0 & 0.999957 & 0.000043 \\ 0.006000 & 0.000022 & 0.993978 \end{bmatrix} \quad (A.41)$$

$$P_{071016} = \begin{bmatrix} 0.970380 & 0 & 0.029620 \\ 0 & 0.999767 & 0.000233 \\ 0.000492 & 0.002183 & 0.997325 \end{bmatrix} \quad (A.42)$$

$$P_{071017} = \begin{bmatrix} 0.999822 & 0 & 0.000178 \\ 0 & 0.984303 & 0.015696 \\ 0.001236 & 0.000532 & 0.998231 \end{bmatrix} \quad (A.43)$$

$$P_{072405} = \begin{bmatrix} 0.999199 & 0 & 0.000801 \\ 0 & 0.998529 & 0.001471 \\ 0.005527 & 0.001480 & 0.992993 \end{bmatrix} \quad (A.44)$$

$$P_{072406} = \begin{bmatrix} 0.968181 & 0 & 0.031819 \\ 0 & 0.999862 & 0.000138 \\ 0.001882 & 0.000336 & 0.997782 \end{bmatrix} \quad (A.45)$$

$$P_{072407} = \begin{bmatrix} 0.999022 & 0 & 0.000978 \\ 0 & 0.982289 & 0.017711 \\ 0.001822 & 0.000461 & 0.997717 \end{bmatrix} \quad (A.46)$$

$$P_{072408} = \begin{bmatrix} 0.999758 & 0 & 0.000242 \\ 0 & 0.972781 & 0.027219 \\ 0.002899 & 0.001175 & 0.995926 \end{bmatrix} \quad (A.47)$$

$$P_{072409} = \begin{bmatrix} 0.946720 & 0 & 0.053280 \\ 0 & 0.999947 & 0.000053 \\ 0.012784 & 0.003153 & 0.984063 \end{bmatrix} \tag{A.48}$$

$$P_{072410} = \begin{bmatrix} 0.924346 & 0 & 0.075654 \\ 0 & 0.999988 & 0.000012 \\ 0.094282 & 0.000144 & 0.905574 \end{bmatrix} \tag{A.49}$$

$$P_{072411} = \begin{bmatrix} 0.785913 & 0 & 0.214087 \\ 0 & 0.999866 & 0.000144 \\ 3.0 \times 10^{-9} & 0.000257 & 0.999743 \end{bmatrix} \tag{A.50}$$

$$P_{072412} = \begin{bmatrix} 0.998567 & 0 & 0.001433 \\ 0 & 0.978809 & 0.021191 \\ 0.000654 & 0.000591 & 0.998755 \end{bmatrix}. \tag{A.51}$$

The transition probabilities for the simplified model that give the best fits to the $\overline{P}(m,n)$ for $\eta = -10$ dB are

$$P_{070901} = \begin{bmatrix} 0.954652 & 0 & 0.045348 \\ 0 & 0.999952 & 0.000048 \\ 0.026468 & 0.007562 & 0.965970 \end{bmatrix} \tag{A.52}$$

$$P_{070903} = \begin{bmatrix} 0.983985 & 0 & 0.016015 \\ 0 & 0.999248 & 0.000752 \\ 0.002134 & 0.000577 & 0.997289 \end{bmatrix} \tag{A.53}$$

$$P_{070905} = \begin{bmatrix} 0.998006 & 0 & 0.001994 \\ 0 & 0.998027 & 0.001973 \\ 0.001803 & 0.000534 & 0.997663 \end{bmatrix} \tag{A.54}$$

$$P_{070906} = \begin{bmatrix} 0.992056 & 0 & 0.007944 \\ 0 & 0.999884 & 0.000116 \\ 0.002998 & 0.000866 & 0.996136 \end{bmatrix} \tag{A.55}$$

$$P_{070907} = \begin{bmatrix} 0.999625 & 0 & 0.000375 \\ 0 & 0.976691 & 0.023309 \\ 0.003408 & 0.001279 & 0.995313 \end{bmatrix} \tag{A.56}$$

$$P_{070912} = \begin{bmatrix} 0.957686 & 0 & 0.042314 \\ 0 & 0.999974 & 0.000026 \\ 0.006537 & 0.000011 & 0.993452 \end{bmatrix} \tag{A.57}$$

$$P_{070914} = \begin{bmatrix} 0.973795 & 0 & 0.026205 \\ 0 & 0.999942 & 0.000058 \\ 0.003679 & 0.000005 & 0.996315 \end{bmatrix} \tag{A.58}$$

$$P_{071016} = \begin{bmatrix} 0.999836 & 0 & 0.000164 \\ 0 & 0.982809 & 0.017191 \\ 0.001639 & 0.000666 & 0.997695 \end{bmatrix} \tag{A.59}$$

$$P_{071017} = \begin{bmatrix} 0.999833 & 0 & 0.000167 \\ 0 & 0.987311 & 0.012689 \\ 0.001813 & 0.000726 & 0.997461 \end{bmatrix} \tag{A.60}$$

$$P_{072405} = \begin{bmatrix} 0.999491 & 0 & 0.000509 \\ 0 & 0.994348 & 0.005652 \\ 0.004751 & 0.0016830 & 0.993565 \end{bmatrix} \tag{A.61}$$

$$P_{072406} = \begin{bmatrix} 0.999983 & 0 & 0.000017 \\ 0 & 0.978613 & 0.021387 \\ 0.000099 & 0.003633 & 0.996268 \end{bmatrix} \tag{A.62}$$

$$P_{072407} = \begin{bmatrix} 0.961790 & 0 & 0.038210 \\ 0 & 0.999169 & 0.000831 \\ 0.002670 & 0.000491 & 0.996839 \end{bmatrix} \tag{A.63}$$

$$P_{072408} = \begin{bmatrix} 0.981025 & 0 & 0.018975 \\ 0 & 0.999916 & 0.000084 \\ 0.006131 & 0.001592 & 0.992275 \end{bmatrix} \quad (A.64)$$

$$P_{072409} = \begin{bmatrix} 0.994727 & 0 & 0.005273 \\ 0 & 0.999980 & 0.000020 \\ 0.015074 & 0.003443 & 0.981483 \end{bmatrix} \quad (A.65)$$

$$P_{072410} = \begin{bmatrix} 0.994882 & 0 & 0.005118 \\ 0 & 0.982182 & 0.017818 \\ 0.002024 & 0.000782 & 0.997193 \end{bmatrix} \quad (A.66)$$

$$P_{072411} = \begin{bmatrix} 0.903834 & 0 & 0.096166 \\ 0 & 0.999975 & 0.000025 \\ 0.067573 & 0.005180 & 0.927247 \end{bmatrix} \quad (A.67)$$

$$P_{072412} = \begin{bmatrix} 0.999952 & 0 & 0.000048 \\ 0 & 0.986267 & 0.013733 \\ 0.000060 & 0.002112 & 0.997828 \end{bmatrix} . \quad (A.68)$$

The transition probabilities for the fully connected model that give the best fits to $\overline{P}(m,n)$ for $\eta = -6$ dB are

$$P_{070901} = \begin{bmatrix} 0.999967 & 0.000016 & 0.000016 \\ 0.001863 & 0.514052 & 0.484084 \\ 0.001840 & 0.514051 & 0.484108 \end{bmatrix} \quad (A.69)$$

$$P_{070903} = \begin{bmatrix} 0.999718 & 0.000146 & 0.000135 \\ 0.000759 & 0.499878 & 0.499362 \\ 0.000702 & 0.491899 & 0.507397 \end{bmatrix} \quad (A.70)$$

$$P_{070905} = \begin{bmatrix} 0.999640 & 0.000212 & 0.000146 \\ 0.000578 & 0.529753 & 0.469667 \\ 0.000539 & 0.470794 & 0.528666 \end{bmatrix} \quad (A.71)$$

$$P_{070906} = \begin{bmatrix} 0.999904 & 0.000047 & 0.000048 \\ 0.000813 & 0.499879 & 0.499307 \\ 0.000833 & 0.496596 & 0.502570 \end{bmatrix} \tag{A.72}$$

$$P_{070907} = \begin{bmatrix} 0.999893 & 0.000078 & 0.000027 \\ 0.001186 & 0.500278 & 0.498535 \\ 0.001174 & 0.485413 & 0.513412 \end{bmatrix} \tag{A.73}$$

$$P_{070912} = \begin{bmatrix} 0.999769 & 000116 & 0.000113 \\ 0.000349 & 0.500826 & 0.498825 \\ 0.000353 & 0.502543 & 0.497103 \end{bmatrix} \tag{A.74}$$

$$P_{070914} = \begin{bmatrix} 0.998076 & 0.001923 & 0.000000 \\ 0.000399 & 0.470270 & 0.529329 \\ 0.000397 & 0.529344 & 0.470258 \end{bmatrix} \tag{A.75}$$

$$P_{071016} = \begin{bmatrix} 0.999961 & .000020 & 0.000019 \\ 0.000685 & 0.519654 & 0.479661 \\ 0.000666 & 0.519653 & 0.479680 \end{bmatrix} \tag{A.76}$$

$$P_{071017} = \begin{bmatrix} 0.999940 & 0.000059 & 0.000000 \\ 0.000455 & 0.519764 & 0.479781 \\ 0.000440 & 0.528401 & 0.471158 \end{bmatrix} \tag{A.77}$$

$$P_{072405} = \begin{bmatrix} 0.999706 & 0.000157 & 0.000136 \\ 0.001348 & 0.517311 & 0.481341 \\ 0.001330 & 0.517311 & 0.481359 \end{bmatrix} \tag{A.78}$$

$$P_{072406} = \begin{bmatrix} 0.999927 & 0.000038 & 0.000035 \\ 0.001400 & 0.501972 & 0.496627 \\ 0.001406 & 0.501297 & 0.497296 \end{bmatrix} \tag{A.79}$$

$$P_{072407} = \begin{bmatrix} 0.999845 & 0.000075 & 0.000079 \\ 0.000396 & 0.498335 & 0.501268 \\ 0.000396 & 0.490799 & 0.508804 \end{bmatrix} \tag{A.80}$$

$$P_{072408} = \begin{bmatrix} 0.999969 & 0.000015 & 0.000015 \\ 0.000285 & 0.520858 & 0.478856 \\ 0.000272 & 0.520857 & 0.478871 \end{bmatrix} \tag{A.81}$$

$$P_{072409} = \begin{bmatrix} 0.999993 & 0.000003 & 0.000003 \\ 0.000131 & 0.495935 & 0.503933 \\ 0.000131 & 0.520731 & 0.479137 \end{bmatrix} \tag{A.82}$$

$$P_{072410} = \begin{bmatrix} 0.999552 & 0.000227 & 0.000221 \\ 0.000498 & 0.498251 & 0.501251 \\ 0.000485 & 0.496759 & 0.502756 \end{bmatrix} \tag{A.83}$$

$$P_{072411} = \begin{bmatrix} 0.999982 & 0.0000090 & 0.000008 \\ 0.001027 & 0.521480 & 0.477492 \\ 0.000996 & 0.521480 & 0.477524 \end{bmatrix} \tag{A.84}$$

$$P_{072412} = \begin{bmatrix} 0.999859 & 0.000071 & 0.000069 \\ 0.000601 & 0.508594 & 0.490805 \\ 0.000601 & 0.508593 & 0.490805 \end{bmatrix}. \tag{A.85}$$

The transition probabilities for the fully connected model that give the best fits to $\overline{P}(m,n)$ for $\eta = -10$ dB are

$$P_{070901} = \begin{bmatrix} 0.999962 & 0.000019 & 0.000018 \\ 0.003801 & 0.523098 & 0.473101 \\ 0.003620 & 0.523099 & 0.473281 \end{bmatrix} \tag{A.86}$$

$$P_{070903} = \begin{bmatrix} 0.999687 & 0.000196 & 0.000117 \\ 0.001242 & 0.492089 & 0.506667 \\ 0.000189 & 0.510737 & 0.489073 \end{bmatrix} \tag{A.87}$$

$$P_{070905} = \begin{bmatrix} 0.999589 & 0.000229 & 0.000181 \\ 0.000829 & 0.523365 & 0.475805 \\ 0.000817 & 0.481853 & 0.517329 \end{bmatrix} \quad (A.88)$$

$$P_{070906} = \begin{bmatrix} 0.999881 & 0.000068 & 0.000049 \\ 0.001448 & 0.503986 & 0.494565 \\ 0.001438 & 0.473038 & 0.525523 \end{bmatrix} \quad (A.89)$$

$$P_{070907} = \begin{bmatrix} 0.999905 & 0.000054 & 0.000040 \\ 0.001261 & 0.509412 & 0.489326 \\ 0.001074 & 0.497855 & 0.501070 \end{bmatrix} \quad (A.90)$$

$$P_{070912} = \begin{bmatrix} 0.998757 & 0.001242 & 0.000000 \\ 0.000315 & 0.434444 & 0.565240 \\ 0.000309 & 0.555178 & 0.444512 \end{bmatrix} \quad (A.91)$$

$$P_{070914} = \begin{bmatrix} 0.998472 & 0.001528 & 0.000000 \\ 0.000348 & 0.468339 & 0.531312 \\ 0.000339 & 0.532906 & 0.466753 \end{bmatrix} \quad (A.92)$$

$$P_{071016} = \begin{bmatrix} 0.999958 & 0.000022 & 0.000020 \\ 0.000809 & 0.521593 & 0.477597 \\ 0.000781 & 0.521591 & 0.477628 \end{bmatrix} \quad (A.93)$$

$$P_{071017} = \begin{bmatrix} 0.999947 & 0.000028 & 0.000024 \\ 0.001280 & 0.525332 & 0.473387 \\ 0.001270 & 0.525332 & 0.473398 \end{bmatrix} \quad (A.94)$$

$$P_{072405} = \begin{bmatrix} 0.999826 & 0.000124 & 0.000049 \\ 0.001218 & 0.522165 & 0.476615 \\ 0.001295 & 0.488179 & 0.510524 \end{bmatrix} \quad (A.95)$$

$$P_{072406} = \begin{bmatrix} 0.999922 & 0.000055 & 0.000022 \\ 0.001843 & 0.497305 & 0.500851 \\ 0.001834 & 0.487497 & 0.510668 \end{bmatrix} \tag{A.96}$$

$$P_{072407} = \begin{bmatrix} 0.999771 & 0.000117 & 0.000111 \\ 0.001072 & 0.503332 & 0.495595 \\ 0.001095 & 0.482973 & 0.515931 \end{bmatrix} \tag{A.97}$$

$$P_{072408} = \begin{bmatrix} 0.999887 & 0.000058 & 0.000054 \\ 0.003220 & 0.517328 & 0.479451 \\ 0.003220 & 0.517329 & 0.479511 \end{bmatrix} \tag{A.98}$$

$$P_{072409} = \begin{bmatrix} 0.999961 & 0.000019 & 0.000019 \\ 0.004879 & 0.498104 & 0.497017 \\ 0.004760 & 0.492643 & 0.502596 \end{bmatrix} \tag{A.99}$$

$$P_{072410} = \begin{bmatrix} 0.999438 & 0.000303 & 0.000258 \\ 0.000991 & 0.515323 & 0.483685 \\ 0.000959 & 0.482155 & 0.516886 \end{bmatrix} \tag{A.100}$$

$$P_{072411} = \begin{bmatrix} 0.999980 & 0.000009 & 0.000010 \\ 0.001741 & 0.508977 & 0.489281 \\ 0.001720 & 0.504130 & 0.494150 \end{bmatrix} \tag{A.101}$$

$$P_{072412} = \begin{bmatrix} 0.999842 & 0.000085 & 0.000072 \\ 0.000738 & 0.496290 & 0.502971 \\ 0.000731 & 0.501208 & 0.498061 \end{bmatrix}. \tag{A.102}$$

Bibliography

[1] L. N. Kanal and A. R. K. Sastry. Models for channels with memory and their applications to error control. *Proceedings of the IEEE*, 66(7):724–744, July 1978.

[2] F. Swarts and H. Ferreira. Markov characterization of digital mobile VHF channels. *IEEE Transactions on Vehicular Technology*, 43(4):977–985, November 1994.

[3] H. Wang and N. Moayeri. Finite-state Markov channel — a useful model for radio communication channels. *IEEE Transactions on Vehicular Technology*, 44(1):163–171, February 1995.

[4] J. Goldhirsh and W. Vogel. Propagation effects for land mobile satellite systems: Overview of experimental and modeling results. Technical report, NASA, 1992.

[5] E.N. Gilbert. Capacity of a burst-noise channel. *The Bell System Technical Journal*, 1960.

[6] E. O. Elliott. Estimates of error rates for codes on burst-noise channels. *Bell Systems Technical Journal*, 42:1977–1997, September 1963.

[7] R. H. McCullough. The binary regenerative channel. *Bell Systems Technical Journal*, 47:1713–1735, October 1968.

[8] B. D. Fritchman. A binary channel characterization using partitioned Markov chains. *IEEE Transactions on Information Theory*, 13:221–227, April 1967.

[9] B. S. Abbe and M. Agan. ACTS mobile terminals. *International Journal of Satellite Communications*, 1996. to appear.

[10] K. Dessouky and T. Jedrey. The ACTS mobile terminal (AMT). In *Proceedings of the AIAA Conference*, Washington, DC, November 1992.

[11] A. Densmore and V. Jamnejad. A satellite-tracking k- and ka-band mobile vehicle antenna system. *IEEE Transactions on Vehicular Technology*, 1993.

[12] M. Rice, D. Pinck, J. Slack, and B. Humpherys. K-band land-mobile satellite channel characterization using ACTS. *International Journal of Satellite Communications*, 1996. to appear.

[13] Brian Humpherys. Statistical models for the acts k-band land mobile satellite communications channel. Master's thesis, Brigham Young University, 1996.

[14] Bruce D. Fritchman. *A Binary Channel Characterization Using Partitioned Markov Chains with Applications to Error Correcting Codes.* PhD thesis, Lehigh University, 1966.